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FREE VIBRATIONS OF CONICAL SHELLS

by William C. L. Hu

Prepared under Contract No. NASr-94(06) *by*
SOUTHWEST RESEARCH INSTITUTE
San Antonio, Texas
for

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SYMBOLS

s	distance along meridian, measured from apex
θ	circumferential coordinate angle
s_1, s_2	distance from apex to the edge of minor and major base, respectively
a, b	radius of the major and minor base, respectively

$x = \log (s_2/s)$, meridional coordinate

$L = \log (s_2/s_1) = \log (a/b)$

$\mathcal{D} = d/dx$, differential operator

α	semivertex angle
h	thickness of shell

$\epsilon = h^2/12a^2$, thickness parameter

ρ	mass density
ν	Poisson's ratio
E	Young's modulus

$C = Eh/(1 - \nu^2)$, extensional modulus

$D = Eh^3/12(1 - \nu^2)$, flexural modulus

κ	shear constant

$k_s = \kappa(1 - \nu)/2$

N_s, N_θ, $N_{s\theta}$	stress resultants
M_s, M_θ, $M_{s\theta}$	stress couple resultants
Q_s, Q_θ	Transverse shearing stress resultants

u, v, w	displacements of middle surface
β_s, β_θ	angular displacements of normal to middle surface
n	circumferential wave number
t	time
ω	circular frequency in rad/sec
$\Omega = \omega a \sqrt{\rho(1 - \nu^2)/E}$,	dimensionless frequency parameter
F, K, H, J	operator matrices
H^{ij}, J^{ij}, etc.	operators, elements at ith row and jth column of operator matrices H, J, etc.

FREE VIBRATIONS OF CONICAL SHELLS

By William C. L. Hu

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SUMMARY

A method is presented for calculating the natural frequencies and associated modes of axisymmetric and nonsymmetric vibrations of truncated conical shells. The effects of transverse shear deformation and rotatory inertia are included in the formulation. The determination of the natural frequencies and mode functions is reduced to the calculation of eigenvalues and associated eigenvectors of a coefficient matrix, whose size depends on the number of terms retained in the Fourier expansions of the mode functions. Numerical examples are given to illustrate the calculation procedure. Axisymmetric vibrations of free-free conical shells are investigated based on a five-term truncation of the Fourier series of the mode functions, with special emphasis on the variation of the frequency spectrum with respect to the semivertex angle and the completeness parameter of the conical shell.

INTRODUCTION

In recent years, a great amount of effort has been exerted by many investigators to determine the natural frequencies and mode shapes of truncated conical shells with various boundary conditions. A review of the literature (Ref. 1) indicated that, due to the difficulty of analytic treatment of the problem, most investigators had to employ energy methods with simple assumed mode functions (Refs. 2-8), while some others resorted to numerical integration (Refs. 9-12) which is less efficient in solving eigenvalue problems than in solving initial value problems. Among prior investigations, Federhofer (Ref. 2) and Saunders, Wisniewski and Paslay (Ref. 6) used truncated power series or polynomials as assumed mode functions, while Grigolyuk (Ref. 3) and Herrmann and Mirsky (Ref. 4) used trigonometric functions as assumed mode shapes, then calculated the frequencies by a Rayleigh-Ritz procedure. Shulman (Ref. 5) studied several approximate approaches and made an extensive comparison of the numerical results, which revealed a wide discrepancy between different methods. Seide (Ref. 7) used a Donnell type energy expression in a Rayleigh-Ritz procedure but neglected the effects of longitudinal inertia in his kinetic energy expression. He expanded the mode functions into infinite series and calculated the natural frequencies by solving the truncated determinantal equation. He remarked that there appear to be no generalizations that can be made for simplifying the calculation of the entire frequency spectrum. In a recent paper, Garnet and Kempner (Ref. 8) studied the axisymmetric vibrations of conical shells by a Rayleigh-Ritz procedure which incorporated the effects of transverse shear deformation and rotatory inertia. As in many other papers, their numerical analysis has to be based on a one- or two-term truncation of the series expansion of the mode functions, which is, in general, too crude to be consistent with their overaccurate energy expressions and strain expressions.

In contrast to the Rayleigh-Ritz approaches, Goldberg, Bogdanoff and Marcus (Ref. 9) solved the axisymmetric vibrations of truncated conical shells by a numerical integration process. They converted the governing equations into a system of six first-order differential equations which contain the unknown frequency in their coefficients, and determined the natural frequencies by trial and error. In a later paper, Goldberg, Bogdanoff and Alspaugh (Ref. 10) extended the technique to the case of nonsymmetric vibrations of conical shells, in which the numerical integration of an eighth-order set of twelve equations has to be carried out five times for each trial value of frequency, and the calculations are repeated until all the boundary conditions can be satisfied with desired accuracy. Recently Kalnins (Refs. 11, 12) developed a more general numerical integration procedure which enables one to calculate the natural frequencies and mode shapes of an arbitrary or multisegmental shell of revolution. Since considerable amount of computer time is required to calculate each natural frequency and mode of a given shell by numerical integration process, it appears unfeasible to apply these methods (Refs. 9-12) to probe the frequency spectra of truncated conical shells with different conicity and completeness parameter.

In the present paper, a method consisting of an operator-matrix technique and a Galerkin procedure is presented for the investigation of free vibrations of truncated conical shells with both edges free, clamped, or simply supported. The effects of transverse shear deformation and rotatory inertia are included in the formulation, and approximate theories, not including transverse shear and rotatory inertia or further neglecting the bending effects, are derived therefrom. The essence of this method is that the determination of natural frequencies and mode shapes is reduced to the calculation of the eigenvalues and eigenvectors of some coefficient matrix, which can be efficiently performed on a digital computer and involves no trial and error. The size of the matrix depends on the number of terms retained in the Fourier expansions of the mode functions. With the help of a computer having adequate storage capacity, it is not difficult to retain five to ten terms in these series.

For the purpose of illustrating the use of the method given in this paper, numerical calculations are carried out, based on the approximate theory for very thin conical shells for which the bending effects are negligible. The variation of the natural frequencies of axisymmetric vibrations with respect to the conicity and the completeness parameter is investigated by a five-term truncation of the Fourier series of the mode functions.

The author wishes to acknowledge his appreciation to Dr. U. S. Lindholm and Dr. W. H. Chu for valuable comments and helpful discussions, and also Mr. R. Gonzales for his assistance in the numerical computations.

ANALYSIS

Matrix Form of Governing Differential Equations

Referred to the curvilinear coordinate system s and θ on the middle surface of the conical shell (Fig. 1), the five equations of motion of the shell element (Fig. 2) which include the effects of transverse shear and rotatory inertia are

$$\frac{\partial N_s}{\partial s} + \frac{N_s}{s} + \frac{1}{s \sin \alpha} \frac{\partial N_{\theta s}}{\partial \theta} - \frac{N_\theta}{s} = \rho h \frac{\partial^2 u}{\partial t^2} \tag{1a}$$

$$\frac{\partial N_{s\theta}}{\partial s} + \frac{2N_{s\theta}}{s} + \frac{1}{s \sin \alpha} \frac{\partial N_\theta}{\partial \theta} + \frac{Q_\theta}{s \tan \alpha} = \rho h \frac{\partial^2 v}{\partial t^2} \tag{1b}$$

$$-\frac{N_\theta}{s \tan \alpha} + \frac{\partial Q_s}{\partial s} + \frac{Q_s}{s} + \frac{1}{s \sin \alpha} \frac{\partial Q_\theta}{\partial \theta} = \rho h \frac{\partial^2 w}{\partial t^2} \tag{1c}$$

$$\frac{\partial M_s}{\partial s} + \frac{M_s}{s} + \frac{1}{s \sin \alpha} \frac{\partial M_{\theta s}}{\partial \theta} - \frac{M_\theta}{s} - Q_s = \frac{1}{12} \rho h^3 \frac{\partial^2 \beta_s}{\partial t^2} \tag{1d}$$

$$\frac{\partial M_{s\theta}}{\partial s} + \frac{2M_{s\theta}}{s} + \frac{1}{s \sin \alpha} \frac{\partial M_\theta}{\partial \theta} - Q_\theta = \frac{1}{12} \rho h^3 \frac{\partial^2 \beta_\theta}{\partial t^2} \tag{1e}$$

The stress-strain relations of thin elastic shells which incorporate transverse shear deformation have been derived by a number of authors using various approaches. The following form of the stress-strain relations, which will be used in this paper, is based on those given by Naghdi (Ref. 13)

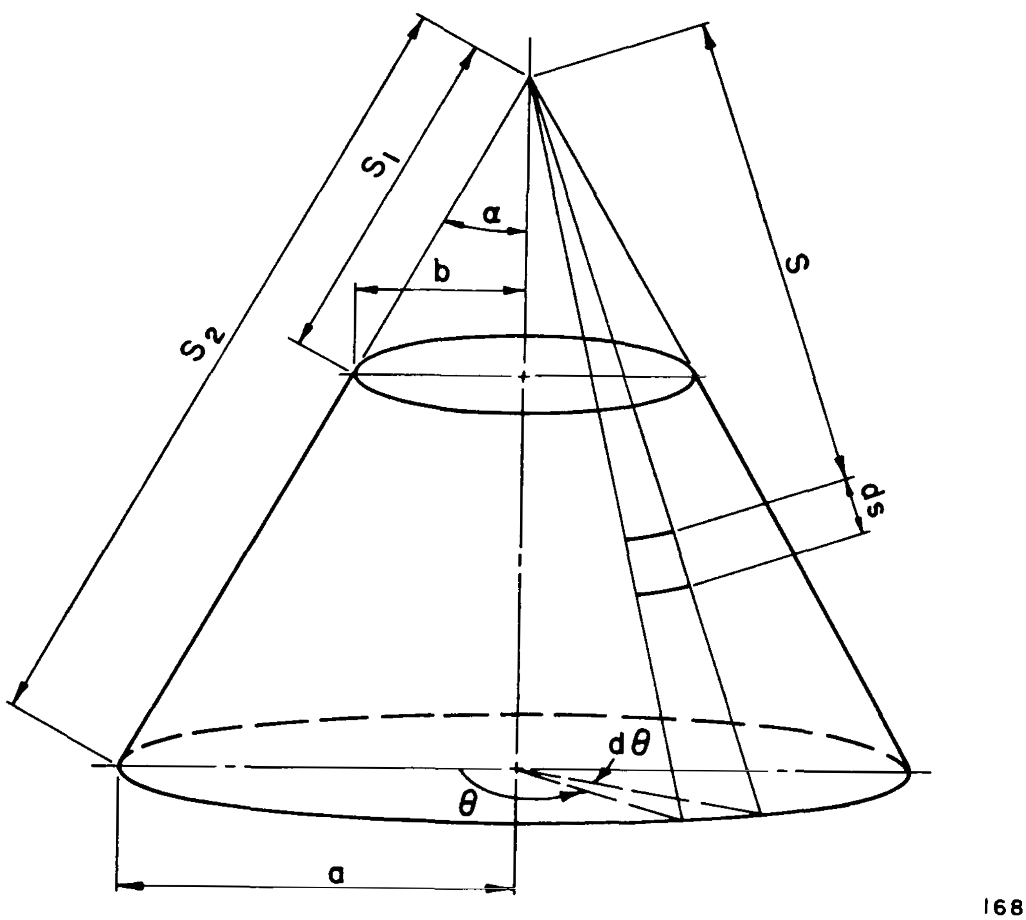
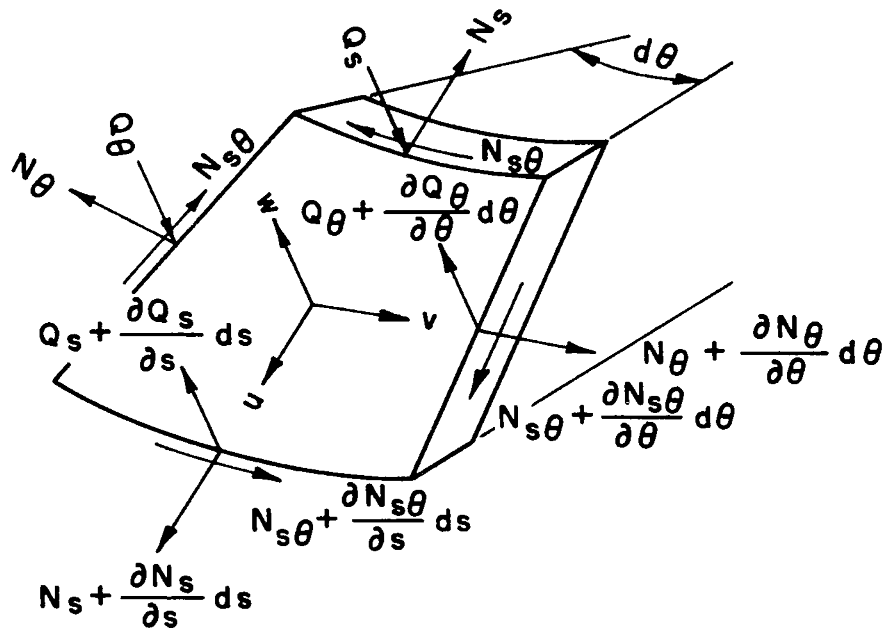
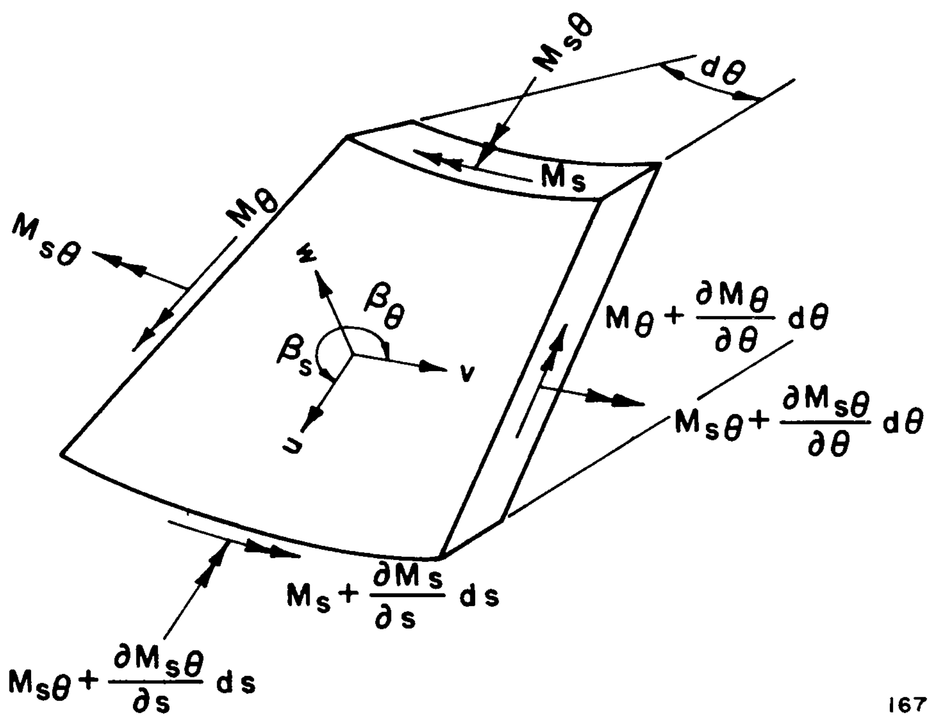


FIGURE 1. GEOMETRY OF MIDDLE SURFACE OF CONICAL SHELL



(a) STRESS RESULTANTS AND DISPLACEMENTS



(b) STRESS COUPLE RESULTANTS AND ROTATIONS

FIGURE 2. STRESSES AND DEFORMATION OF A DIFFERENTIAL ELEMENT

(sometimes referred to as Reissner-Naghdi's theory). By introducing an approximation similar to one used by Naghdi and Cooper (Ref. 14) in their study of cylindrical shells (System II, Ref. 14), these relations for conical shells are

$$N_s = C\left[\frac{\partial u}{\partial s} + \nu\left(\frac{1}{s \sin \alpha}\frac{\partial v}{\partial \theta} + \frac{u}{s} + \frac{w}{s \tan \alpha}\right)\right] \tag{2a}$$

$$N_{s\theta} = N_{\theta s} = \frac{1-\nu}{2} C\left[\frac{\partial v}{\partial s} + \frac{1}{s \sin \alpha}\frac{\partial u}{\partial \theta} - \frac{v}{s}\right] \tag{2b}$$

$$N_\theta = C\left[\frac{1}{s \sin \alpha}\frac{\partial v}{\partial \theta} + \frac{u}{s} + \frac{w}{s \tan \alpha} + \nu\frac{\partial u}{\partial s}\right] \tag{2c}$$

$$M_s = D\left[\frac{\partial \beta_s}{\partial s} + \nu\left(\frac{1}{s \sin \alpha}\frac{\partial \beta_\theta}{\partial \theta} + \frac{\beta_s}{s}\right)\right] \tag{2d}$$

$$M_{s\theta} = M_{\theta s} = \frac{1-\nu}{2} D\left[\frac{\partial \beta_\theta}{\partial s} + \frac{1}{s \sin \alpha}\frac{\partial \beta_s}{\partial \theta} - \frac{\beta_\theta}{s}\right] \tag{2e}$$

$$M_\theta = D\left[\frac{1}{s \sin \alpha}\frac{\partial \beta_\theta}{\partial \theta} + \frac{\beta_s}{s} + \nu\frac{\partial \beta_s}{\partial s}\right] \tag{2f}$$

$$Q_s = \kappa\frac{1-\nu}{2} C\left[\frac{\partial w}{\partial s} + \beta_s\right] \tag{2g}$$

$$Q_\theta = \kappa\frac{1-\nu}{2} C\left[\frac{1}{s \sin \alpha}\frac{\partial w}{\partial \theta} + \beta_\theta\right] \tag{2h}$$

where $C = Eh/(1-\nu^2)$, $D = Eh^3/12(1-\nu^2)$, and κ is a shear coefficient which has the value 5/6 in Reference 13 as a consequence of the consistent assumptions for stresses and displacements. Slightly different values of κ have been obtained by other authors through different considerations*, but the

*For example, in their studies of cylindrical shells, Herrmann and Mirsky (Ref. 15) have used $\kappa = 0.86$ and later (Ref. 16) $\pi^2/12$, Lin and Morgan (Ref. 17) have used 8/9.

effects of this difference are believed to be very small for all practical purposes. It should be noted that, when the effects of rotatory inertia and transverse shear deformation are neglected, the system of governing equations, (1) and (2), reduces to that of the classical bending theory (Ref. 18). If the term $Q_\theta/(s \tan \alpha)$ in (1b) is further neglected, it reduces to the Donnell type theory as given by Seide (Ref. 19).

To satisfy the periodical property in the circumferential direction, the thirteen variables in (1) and (2) are assumed to be separable in the form:

$$\left\{\begin{array}{l} u \\ w \\ \beta_s \\ N_s \\ N_\theta \\ M_s \\ M_\theta \\ Q_s \end{array}\right\} = \left\{\begin{array}{r} a u_n \\ a w_n \\ \beta_{sn} \\ C N_{sn} \\ C N_{\theta n} \\ (D/a) M_{sn} \\ (D/a) M_{\theta n} \\ (D/a^2) Q_{sn} \end{array}\right\} \sin(n\theta + \theta_o) \cos \omega t, \tag{3a}$$

$$\left\{\begin{array}{l} v \\ \beta_\theta \\ N_{s\theta} \\ M_{s\theta} \\ Q_\theta \end{array}\right\} = \left\{\begin{array}{r} a v_n \\ \beta_{\theta n} \\ C N_{s\theta n} \\ (D/a) M_{s\theta n} \\ (D/a^2) Q_{\theta n} \end{array}\right\} \cos(n\theta + \theta_o) \cos \omega t \tag{3b}$$

where ω is a natural frequency, n an integer representing the circumferential wave number, and θ_o a phase angle which is introduced for the convenience of discussion for $n = 0$. Note that as a result of using the reference length $a = s_2 \sin \alpha$ (the radius of the major base), the mode functions with subscript n are dimensionless quantities.

Substitution of (3) into (1) and (2) results in

$$\frac{dN_{sn}}{ds} + \frac{N_{sn}}{s} - \frac{nN_{s\theta n}}{s \sin \alpha} - \frac{N_{\theta n}}{s} = - \frac{\omega^2 \rho (1 - \nu^2) a}{E} u_n \tag{4a}$$

$$\frac{dN_{s\theta n}}{ds} + \frac{2N_{s\theta n}}{s} + \frac{nN_{\theta n}}{s \sin \alpha} + \frac{h^2}{12a^2} \frac{Q_{\theta n}}{s \tan \alpha} = - \frac{\omega^2 \rho (1 - \nu^2) a}{E} v_n \tag{4b}$$

$$- \frac{N_{\theta n}}{s \tan \alpha} + \frac{h^2}{12a^2}\left(\frac{dQ_{sn}}{ds} + \frac{Q_{sn}}{s} - \frac{nQ_{\theta n}}{s \sin \alpha}\right) = - \frac{\omega^2 \rho (1 - \nu^2) a}{E} w_n \quad (4c)$$

$$\frac{dM_{sn}}{ds} + \frac{M_{sn}}{s} - \frac{nM_{s\theta n}}{s \sin \alpha} - \frac{M_{\theta n}}{s} - \frac{Q_{sn}}{a} = - \frac{\omega^2 \rho (1 - \nu^2) a}{E} \beta_{sn} \quad (4d)$$

$$\frac{dM_{s\theta n}}{ds} + \frac{2M_{s\theta n}}{s} + \frac{nM_{\theta n}}{s \sin \alpha} - \frac{Q_{\theta n}}{a} = - \frac{\omega^2 \rho (1 - \nu^2) a}{E} \beta_{\theta n} \quad (4e)$$

$$N_{sn} = a \frac{du_n}{ds} + \nu a \left(- \frac{nv_n}{s \sin \alpha} + \frac{u_n}{s} + \frac{w_n}{s \tan \alpha}\right) \quad (5a)$$

$$N_{s\theta n} = \frac{1 - \nu}{2} a \left(\frac{dv_n}{ds} + \frac{nu_n}{s \sin \alpha} - \frac{v_n}{s}\right) \quad (5b)$$

$$N_{\theta n} = - \frac{nav_n}{s \sin \alpha} + \frac{au_n}{s} + \frac{aw_n}{s \tan \alpha} + \nu a \frac{du_n}{ds} \quad (5c)$$

$$M_{sn} = a \frac{d\beta_{sn}}{ds} + \nu a \left(- \frac{n\beta_{\theta n}}{s \sin \alpha} + \frac{\beta_{sn}}{s}\right) \quad (5d)$$

$$M_{s\theta n} = \frac{1 - \nu}{2} a \left(\frac{d\beta_{\theta n}}{ds} + \frac{n\beta_{sn}}{s \sin \alpha} - \frac{\beta_{\theta n}}{s}\right) \quad (5e)$$

$$M_{\theta n} = - \frac{an\beta_{\theta n}}{s \sin \alpha} + \frac{a\beta_{sn}}{s} + \nu a \frac{d\beta_{sn}}{ds} \quad (5f)$$

$$Q_{sn} = \frac{\kappa(1 - \nu)}{2} \frac{12a^2}{h^2}\left(\frac{adw_n}{ds} + \beta_{sn}\right) \quad (5g)$$

$$Q_{\theta n} = \frac{\kappa (1 - \nu)}{2} \frac{12a^2}{h^2}\left(\frac{anw_n}{s \sin \alpha} + \beta_{\theta n}\right) \quad (5h)$$

The thirteen differential equations (4) and (5) contain thirteen unknowns but involve only ten of their first derivatives, since the first derivatives of the three variables N_{sn}, M_{sn}, and Q_{sn} do not appear. Therefore, the system is of tenth order, and requires five appropriate boundary conditions at each edge to ensure the problem completely determinate.

Introducing the well-known coordinate transformation for conical shells,

$$s = s_2 e^{-x} \tag{6}$$

and noting that the interval $s_1 \leqslant s \leqslant s_2$ is transformed to $0 \leqslant x \leqslant L$, with

$$L = \log\left(\frac{s_2}{s_1}\right) = \log\left(\frac{a}{b}\right) \tag{7}$$

we can write the equations of motion, (4), in the matrix form:

$$F \begin{Bmatrix} N_{sn} \\ N_{s\theta n} \\ N_{\theta n} \\ M_{sn} \\ M_{s\theta n} \\ M_{\theta n} \\ Q_{sn} \\ Q_{\theta n} \end{Bmatrix} = \Omega^2 \begin{Bmatrix} u_n \\ v_n \\ w_n \\ \beta_{sn} \\ \beta_{\theta n} \end{Bmatrix} \tag{8}$$

where Ω is the dimensionless natural frequency parameter (hereafter simply referred to as frequency when no confusion will arise).

$$\Omega = \omega a \sqrt{\rho(1 - \nu^2)/E} \tag{9}$$

and F is the (5×8) operator matrix:

$$
F = e^{x} \sin \alpha \begin{bmatrix}
\mathcal{D}-1 & \dfrac{n}{\sin \alpha} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathcal{D}-2 & \dfrac{-n}{\sin \alpha} & 0 & 0 & 0 & 0 & -\epsilon \cot \alpha \\
0 & 0 & \cot \alpha & 0 & 0 & 0 & \epsilon(\mathcal{D}-1) & \dfrac{\epsilon n}{\sin \alpha} \\
0 & 0 & 0 & \mathcal{D}-1 & \dfrac{n}{\sin \alpha} & 1 & \dfrac{e^{-x}}{\sin \alpha} & 0 \\
0 & 0 & 0 & 0 & \mathcal{D}-2 & \dfrac{-n}{\sin \alpha} & 0 & \dfrac{e^{-x}}{\sin \alpha}
\end{bmatrix} \tag{10}
$$

in which $\mathcal{D} = d/dx$ is the differential operator and $\epsilon = h^2/12a^2$ the thickness parameter. Similarly, the stress-strain relations (5) are transformed to the matrix form:

$$
\begin{Bmatrix} N_{sn} \\ N_{s\theta n} \\ N_{\theta n} \\ M_{sn} \\ M_{s\theta n} \\ M_{\theta n} \\ Q_{sn} \\ Q_{\theta n} \end{Bmatrix} = K \begin{Bmatrix} u_n \\ v_n \\ w_n \\ \beta_{sn} \\ \beta_{\theta n} \end{Bmatrix} \tag{11}
$$

where K is the (8 × 5) operator matrix:

$$K = e^x \sin\alpha \begin{bmatrix} -\mathcal{D}+\nu & \frac{\nu n}{\sin\alpha} & \nu\cot\alpha & 0 & 0 \\ \frac{1-\nu}{2}\frac{n}{\sin\alpha} & -\frac{1-\nu}{2}(\mathcal{D}+1) & 0 & 0 & 0 \\ -\nu\mathcal{D}+1 & -\frac{n}{\sin\alpha} & \cot\alpha & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{D}+\nu & -\frac{\nu n}{\sin\alpha} \\ 0 & 0 & 0 & \frac{1-\nu}{2}\frac{n}{\sin\alpha} & -\frac{1-\nu}{2}(\mathcal{D}+1) \\ 0 & 0 & 0 & -\nu\mathcal{D}+1 & -\frac{n}{\sin\alpha} \\ 0 & 0 & -\frac{k_s}{\epsilon}\mathcal{D} & \frac{k_s}{\epsilon}\frac{e^{-x}}{\sin\alpha} & 0 \\ 0 & 0 & \frac{k_s}{\epsilon}\frac{n}{\sin\alpha} & 0 & \frac{k_s}{\epsilon}\frac{e^{-x}}{\sin\alpha} \end{bmatrix} \tag{12}$$

in which $k_s = \kappa(1-\nu)/2$.

Let N denote the stress column vector:

$$N = \left\{ N_{sn}\, N_{s\theta n}\, N_{\theta n}\, M_{sn}\, M_{s\theta n}\, M_{\theta n}\, Q_{sn}\, Q_{\theta n} \right\},$$

and U denote the displacement column vector

$$U = \left\{ u_n\, v_n\, w_n\, \beta_{sn}\, \beta_{\theta n} \right\}$$

then, from the matrix equations (8) and (11), we can eliminate either N or U to get

$$HU = \Omega^2 U \tag{13}$$

or

$$JN = \Omega^2 N \tag{14}$$

where H is the (5×5) operator matrix

$$H = FK$$

and J is the (8×8) operator matrix

$$J = KF$$

To simplify the multiplication of two operator matrices, we observe the following rules from operator calculus:

(1) Let $P(\mathcal{D})$ be any polynomial of the differential operator $\mathcal{D} = d/dx$, then the operator $\cdot e^x$ obeys the shifting rule

$$P(\mathcal{D})\, e^x = e^x P(\mathcal{D} + 1)$$

and, in general, for any constant c,

$$P(\mathcal{D})\, e^{cx} = e^{cx} P(\mathcal{D} + c)$$

This rule also applies to operator matrices whose elements contain polynomials of $\mathcal{D}$. Thus, for example, if $[Q(\mathcal{D})]$ and $[Q'(\mathcal{D})]$ are two such operator matrices, we have

$$[Q(\mathcal{D})] \cdot e^x [Q'(\mathcal{D})] = e^x \cdot [Q(\mathcal{D} + 1)] \cdot [Q'(\mathcal{D})]$$

(2) Let c and c' be arbitrary constants, then

$$(\mathcal{D} + c)(\mathcal{D} + c') = \mathcal{D}^2 + (c + c')\,\mathcal{D} + cc'$$

With the help of these rules, we can easily calculate the two operator matrices H and J:

$$H = [H^{ij}], \qquad i, j = 1, 2, \ldots, 5$$

and

$$J = [J^{ij}], \qquad i, j = 1, 2, 3, \ldots, 8$$

The operators H^{ij} and J^{ij} are given in Appendix I.

The remaining procedures of solution of the vibration problem depend on the boundary conditions prescribed at the two circular edges, i.e., at $x = 0, L$. It will be seen that (14) is the more convenient form for free-free conical shells, while (13) is better suited for conical shells with both edges clamped or simply supported. In the following a few important edge conditions are considered separately.

Calculations of Frequencies and Modes

Clamped edges. - When the truncated conical shell is clamped at both edges, the boundary conditions are, at $x = 0, L$,

$$u_n = v_n = w_n = \beta_{sn} = \beta_{\theta n} = 0$$

Therefore, the five mode functions can be expanded into sine series, every term of which satisfies the boundary conditions independently. After truncation, they are

$$u_n = \sum_{m=1}^{M_1} a_m \sin \frac{m\pi x}{L} \tag{15a}$$

$$v_n = \sum_{m=1}^{M_2} b_m \sin \frac{m\pi x}{L} \tag{15b}$$

$$w_n = \sum_{m=1}^{M_3} c_m \sin \frac{m\pi x}{L} \tag{15c}$$

$$\beta_{sn} = \sum_{m=1}^{M_4} d_m \sin \frac{m\pi x}{L} \tag{15d}$$

$$\beta_{\theta n} = \sum_{m=1}^{M_5} e_m \sin \frac{m\pi x}{L} \tag{15e}$$

where $M_1, \ldots, M_5$ are properly chosen integers representing the number of terms to be retained in the series, and which may be taken equal to each other. Denoting $M = M_1 + M_2 + M_3 + M_4 + M_5$, there are M undetermined constants in the problem besides the unknown frequency.

We assume that the Fourier expansions of the five mode functions are termwise differentiable twice.* Substitition of (15) into (13) results in five equations containing the M coefficients. The first equation, for example, can be written as:

$$\sum_{m=1}^{M_1} a_m \left(H^{11} \sin \frac{m\pi x}{L}\right) + \sum_{m=1}^{M_2} b_m \left(H^{12} \sin \frac{m\pi x}{L}\right) + \sum_{m=1}^{M_3} c_m \left(H^{13} \sin \frac{m\pi x}{L}\right)$$

$$+ \sum_{m=1}^{M_4} d_m \left(H^{14} \sin \frac{m\pi x}{L}\right) + \sum_{m=1}^{M_5} e_m \left(H^{15} \sin \frac{m\pi x}{L}\right) = \Omega^2 \sum_{k=1}^{M_1} a_k \sin \frac{k\pi x}{L} \tag{16}$$

where the operators H^{1j}, $j = 1, 2, \ldots, 5$, are, as given in Appendix I,

$$H^{11} = e^{2x} \sin^2 \alpha (-\mathcal{D}^2 + 1) + \frac{1 - \nu}{2} ne^{2x}$$

$$H^{12} = -e^{2x} n \sin \alpha \left(\frac{1 + \nu}{2} \mathcal{D} + \frac{3 - \nu}{2}\right)$$

$$H^{13} = e^{2x} \sin \alpha \cos \alpha (\nu \mathcal{D} + 1)$$

$$H^{14} = H^{15} = 0.$$

To apply the Galerkin method, we multiply (16) by $\sin \frac{k\pi x}{L}$, $k = 1, 2, \ldots, M_1$, then integrate over the interval $0 \leqslant x \leqslant L$. This gives M_1 equations:

$$\sum_{m=1}^{M_1} a_m H_{km}^{11} + \sum_{m=1}^{M_2} b_m H_{km}^{12} + \sum_{m=1}^{M_3} c_m H_{km}^{13} = \Omega^2 a_k, \qquad k = 1, 2, \ldots, M_1,$$

---

*Let $f(x)$ be a smooth, single-valued function of x in the interval $0 \leqslant x \leqslant L$, then its Fourier sine expansion is usually termwise differentiable twice if $f(0) = f(L) = 0$, and its Fourier cosine expansion is usually termwise differentiable twice if $f'(0) = f'(L) = 0$.

where

$$H^{1j}_{km} = \frac{2}{L}\int_0^L \sin\frac{k\pi x}{L}\left(H^{1j}\sin\frac{m\pi x}{L}\right)dx, \qquad j = 1, 2, 3.$$

Note H^{14}_{km} and H^{15}_{km} are dropped because they are identically zero. By applying the same procedure to all five equations in (13), we get M algebraic equations for the M coefficients, which can be put in the matrix form

$$\begin{array}{rc} & \begin{array}{ccccc} \overbrace{\quad\;\;}^{M_1} & \overbrace{\quad\;\;}^{M_2} & \overbrace{\quad\;\;}^{M_3} & \overbrace{\quad\;\;}^{M_4} & \overbrace{\quad\;\;}^{M_5} \end{array} \\ \begin{array}{r} M_1 \{ \\ M_2 \{ \\ M_3 \{ \\ M_4 \{ \\ M_5 \{ \end{array} & \left[\begin{array}{ccccc} [H^{11}_{km}] & [H^{12}_{km}] & [H^{13}_{km}] & [H^{14}_{km}] & [H^{15}_{km}] \\ [H^{21}_{km}] & [H^{22}_{km}] & [H^{23}_{km}] & [H^{24}_{km}] & [H^{25}_{km}] \\ [H^{31}_{km}] & [H^{32}_{km}] & [H^{33}_{km}] & [H^{34}_{km}] & [H^{35}_{km}] \\ [H^{41}_{km}] & [H^{42}_{km}] & [H^{43}_{km}] & [H^{44}_{km}] & [H^{45}_{km}] \\ [H^{51}_{km}] & [H^{52}_{km}] & [H^{53}_{km}] & [H^{54}_{km}] & [H^{55}_{km}] \end{array}\right] \end{array} \begin{Bmatrix} a_1 \\ \vdots \\ b_1 \\ \vdots \\ c_1 \\ \vdots \\ d_1 \\ \vdots \\ e_1 \\ \vdots \end{Bmatrix} = \Omega^2 \begin{Bmatrix} a_1 \\ \vdots \\ b_1 \\ \vdots \\ c_1 \\ \vdots \\ d_1 \\ \vdots \\ e_1 \\ \vdots \end{Bmatrix} \qquad (17)$$

It is important to note that, for conical shells, the coefficients H^{ij}_{km} can be integrated in closed form. Equation (17) is a standard form of matrix eigenvalue problem. The eigenvalues and corresponding eigenvectors of the $(M \times M)$ matrix $[H^{ij}_{km}]$, which can be readily calculated on a digital computer, give the frequencies and corresponding mode shapes.

Simply supported edges with axial constraint. - The term simple support is originated from the theory of beams and plates. When it is applied to shells, ambiguity may easily arise in concern with axial constraint. If the supports are provided by attaching light but very rigid rings to the edges, then no appreciable axial constraint exists for the cases $n = 0$ and $n = 1$ (neglecting the inertia of the rings), while considerable axial constraint will exist for $n \geqslant 2$. For the case with complete axial constraint, the boundary conditions are, at $x = 0, L$,

$$u_n = v_n = w_n = M_{sn} = \beta_{\theta n} = 0. \qquad (18)$$

From (11) and (12), the condition $M_{sn} = 0$ can be replaced by

$$\frac{d\beta_{sn}}{dx} - \nu\beta_{sn} = 0, \qquad \text{at } x = 0, L$$

Therefore, to satisfy all the boundary conditions term by term, u_n, v_n, w_n, and $\beta_{\theta n}$ are expanded into sine series as before, while β_{sn} must be expanded into cosine series which, after truncation, is

$$\beta_{sn} = e^{\nu x}\left[\frac{d_0}{2} + \sum_{m=1}^{M_4} d_m \cos\frac{m\pi x}{L}\right] \tag{19}$$

By applying the same Galerkin procedure as in the clamped-clamped case, we can obtain an $(M + 1)$ by $(M + 1)$ coefficient matrix, whose eigenvalues and eigenvectors give the frequencies and mode functions of the simply supported conical shell. The calculation can be simplified by using a new variable $B_{sn} = e^{-\nu x}\beta_{sn}$, and transforming (17) accordingly. If the Poisson's effect on boundary conditions is neglected, the factor $e^{\nu x}$ in (19) may be dropped.

Simply supported edges without meridional constraint. - If the restraint on meridional displacements at the two circular edges is completely released, the boundary conditions $u_n = 0$ in (18) should be replaced by $N_{sn} = 0$, at $x = 0, L$. From (11) and (12), these conditions can be written

$$\frac{du_n}{dx} - \nu u_n = 0, \qquad \text{at } x = 0, L$$

Therefore, in addition to taking (19) for β_{sn}, u_n should be taken as

$$u_n = e^{\nu x}\left[\frac{a_0}{2} + \sum_{m=1}^{M_1} a_m \cos\frac{m\pi x}{L}\right]$$

By applying the same Galerkin procedure, we can obtain an $(M + 2)$ by $(M + 2)$ coefficient matrix. The calculation can be simplified by using a new variable $U_n = e^{-\nu x}u_n$, and transforming (17) accordingly. If the Poisson's effect on boundary conditions is neglected, the factor $e^{\nu x}$ above may also be dropped.

Simply supported edges without circumferential shearing constraint. - If the restraint on circumferential displacements at the two circular edges is released, the boundary conditions $v_n = 0$ in (18) should be replaced by $N_{s\theta n} = 0$, at $x = 0, L$. From (11) and (12), these conditions can be written

$$\frac{dv_n}{dx} + v_n = 0, \qquad \text{at } x = 0, L$$

The following expansion will satisfy these conditions:

$$v_n = e^{-x}\left[\frac{b_0}{2} + \sum_{m=1}^{M_2} b_m \cos\frac{m\pi x}{L}\right]$$

The calculation can be simplified by using a new variable $V_n = e^x v_n$, and transforming (17) accordingly.

Free edges. - For a truncated conical shell with both edges free, the boundary conditions are, at $x = 0, L$,

$$N_{sn} = N_{s\theta n} = Q_{sn} = M_{sn} = M_{s\theta n} = 0$$

It is natural that these mode functions should be expanded into sine series and (14) should be employed to solve the problem. Since, as mentioned before, the system of governing differential equations is of tenth order, no boundary conditions need be imposed on the other three variables, namely, $N_{\theta n}$, $M_{\theta n}$, and $Q_{\theta n}$. These three functions assume the role of parameters in the problem and can be expanded either into sine series or cosine series. However, an examination of the operators J^{ij} in Appendix I indicates that the first derivatives of these functions are involved, hence their series expansions must be termwise differentiable to ensure convergence of the Galerkin procedure. Since, in general, $N_{\theta n}$, $M_{\theta n}$, $Q_{\theta n}$ and their derivatives do not vanish at the boundary, their cosine expansions represent even periodical functions, which are continuous for all values of x (with piecewise continuous first derivatives), while their sine expansions represent odd periodical functions with discontinuities at $x = 0, L, 2L$, etc. Therefore, the cosine expansion is preferable to the sine expansion, for the former satisfies a set of sufficient conditions (Ref. 20) which ensures termwise differentiability, while the latter does not. Besides, for functions having nonzero value at $x = 0, L$, the cosine series expansions converge more rapidly than the sine expansions.

Now substituting the five sine series and the three cosine series into (14) and applying the Galerkin procedure, we can readily obtain the coefficient matrix for free-free conical shells, similar to (17). However, the size of the resulting matrix is considerably larger than in the four previous cases.

Transverse Shear Theory for Short Shells

A physical reason for the derivation of the following theory can be drawn from the fact concluded by many prior works (e.g., Refs. 8, 14) that while the effects of rotatory inertia are in general negligibly small, the effects of transverse shear deformation may not be neglected for relatively short shells. It will be postulated that in the vibrations of short conical shells, only the transverse shear deformation in the $s - z$ plane is needed for significant correction provided that the circumferential wave number n is not very large.

It is a well known fact that, when the effects of transverse shear and rotatory inertia in the foregoing theory are neglected, the order of the system of governing differential equations (4) and (5) reduces from ten to eight. In this section, it will be shown that the neglect of transverse shear deformation in the circumferential direction alone will result in this reduction of order. In the following derivation, the circumferential rotatory inertia term $\partial^2\beta_{\theta n}/\partial t^2$ is also neglected for simplicity. However, it is easy to see that the retention of this term does not affect the order of the differential equations. The above discussion can be easily generalized to arbitrary shells of revolution.

If the rotatory inertia term in (1e) is neglected and the transverse shear deformation in (2h) is set to zero, (4e) and (5h) should be replaced, respectively, by

$$Q_{\theta n} = a\left(\frac{dM_{s\theta n}}{ds} + \frac{2M_{s\theta n}}{s} + \frac{nM_{\theta n}}{s \sin \alpha}\right) \tag{4e'}$$

$$\beta_{\theta n} = -\frac{anw_n}{s \sin \alpha} \tag{5h'}$$

Substitution of (4e') into (4b, c) and (5h') into (5d, e, f) gives

$$a\frac{dN_{s\theta n}}{ds} + \frac{2aN_{s\theta n}}{s} + \frac{anN_{\theta n}}{s \sin \alpha} + \frac{\epsilon a^2}{s \tan \alpha}\left(\frac{dM_{s\theta n}}{ds} + \frac{2M_{s\theta n}}{ds} + \frac{nM_{\theta n}}{s \sin \alpha}\right) = -\Omega^2 v_n \tag{4b'}$$

$$-\frac{aN_{\theta n}}{s \tan \alpha} + \epsilon a\left(\frac{dQ_{sn}}{ds} + \frac{Q_{sn}}{s}\right) - \frac{\epsilon a^2 n}{s \sin \alpha}\left(\frac{dM_{s\theta n}}{ds} + \frac{2M_{s\theta n}}{s} + \frac{nM_{\theta n}}{s \sin \alpha}\right) = -\Omega^2 w_n \tag{4c'}$$

$$M_{sn} = \frac{ad\beta_{sn}}{ds} + \nu\left(\frac{a^2n^2w_n}{s^2 \sin^2 \alpha} + \frac{a\beta_{sn}}{s}\right) \tag{5d'}$$

$$M_{s\theta n} = \frac{1-\nu}{2}\left(-\frac{a^2 n}{s \sin \alpha}\frac{dw_n}{ds} + \frac{2a^2 n w_n}{s^2 \sin \alpha} + \frac{a n \beta_{sn}}{s \sin \alpha}\right) \tag{5e'}$$

$$M_{\theta n} = \frac{a^2 n^2 w_n}{s^2 \sin^2 \alpha} + \nu a \frac{d\beta_{sn}}{ds} + \frac{a \beta_{sn}}{s} \tag{5f'}$$

It is seen that the derivative $d\beta_{\theta n}/ds$ is eliminated from the system, and that the three derivatives $dN_{s\theta n}/ds$, $dM_{s\theta n}/ds$ and dQ_{sn}/ds can be combined into two by introducing two new variables, $H_s = CH_{sn}\cos(n\theta + \theta_o) \cdot \cos \omega t$ and $V_s = (D/a^2)\, V_{sn} \sin(n\theta + \theta_o) \cos \omega t$, defined by*

$$H_s = N_{s\theta} + \frac{M_{s\theta}}{s \tan \alpha} \tag{20}$$

$$V_s = Q_s + \frac{1}{s \sin \alpha}\frac{\partial M_{s\theta}}{\partial \theta} \tag{21}$$

or, after using the assumption (3),

$$H_{sn} = N_{s\theta n} + \epsilon a \frac{M_{s\theta n}}{s \tan \alpha} \tag{20a}$$

$$V_{sn} = Q_{sn} - \frac{a n M_{s\theta n}}{s \sin \alpha} \tag{21a}$$

Therefore, the system (4a, b', c', d) and (5a, b, c, d', e', f', g) is degenerated from tenth to eighth order, and requires only four boundary conditions at each edge. Following the same steps as in deriving (13), a matrix equation of the displacement vector $U^* = \left\{u_n \; v_n \; w_n \; \beta_{sn}\right\}$ can be obtained:

$$H^* U^* = \Omega^2 U^* \tag{22}$$

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*These two variables have been used in Ref. 10 and, in a more general form, in Refs. 11, 12. In the case of a plate (either $\alpha \to 90°$, circular, or $\alpha \to 0$, $s \sin \alpha = a \to \infty$, rectangular), Eq. (20) becomes trivial while (21) has been discussed by Kirchhoff, see Ref. 21.

where H^* is a (4×4) operator matrix. The elements of H^* are given in Appendix II.

Equation (22) can be used in place of (13) for conical shells with both edges clamped or simply supported. The remaining steps are the same as described before except that the boundary conditions on $\beta_{\theta n}$ or $M_{s\theta n}$ are now discarded. A similar matrix equation of the stress column vector $N^* = \left\{ N_{sn}\ H_{sn}\ N_{\theta n}\ M_{sn}\ M_{s\theta n}\ M_{\theta n}\ V_{sn} \right\}$ can also be obtained, but will not be considered here.

Bending Theory

As mentioned before, the system of differential equations (1) and (2) reduces to that of the classical bending theory of thin shells if the effects of transverse shear and rotatory inertia are neglected. This can be achieved by neglecting the right-hand-side terms of (1d, e) and setting to zero the quantities in (2g, h) that represent the transverse shear deformation. Thus, in addition to replacing (4e) by (4e'), (5h) by (5h'), (4d) and (5g) should also be replaced respectively by

$$Q_{sn} = a\left(\frac{dM_{sn}}{ds} + \frac{M_{sn}}{s} - \frac{nM_{s\theta n}}{s \sin \alpha} - \frac{M_{\theta n}}{s} \right) \tag{4d'}$$

$$\beta_{sn} = - a \frac{dw_n}{ds} \tag{5g'}$$

After elimination of Q's and β's by using (4d', e') and (5g', h'), and introduction of the coordinate transformation (6), and the three equations of motion (1a, b, c) can be written in matrix form:

$$\overline{F} \begin{Bmatrix} N_{sn} \\ N_{s\theta n} \\ N_{\theta n} \end{Bmatrix} + \widetilde{F} \begin{Bmatrix} M_{sn} \\ M_{s\theta n} \\ M_{\theta n} \end{Bmatrix} = \Omega^2 \begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} \tag{23}$$

where $\overline{F}$ and $\widetilde{F}$ are (3×3) operator matrices,

$$\overline{F} = e^x \begin{bmatrix} \sin \alpha (\mathcal{D} - 1) & n & \sin \alpha \\ 0 & \sin \alpha (\mathcal{D} - 2) & -n \\ 0 & 0 & \cos \alpha \end{bmatrix} \tag{24}$$

$$\widetilde{F} = \epsilon e^{2x} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin\alpha\cos\alpha(\mathcal{D}-2) & -n\cos\alpha \\ \sin^2\alpha(-\mathcal{D}^2+\mathcal{D}) & -2n\sin\alpha(\mathcal{D}-1) & -\sin^2\alpha\,\mathcal{D}+n^2 \end{bmatrix} \tag{25}$$

and the stress-strain relations (2a - f) become

$$\begin{Bmatrix} N_{sn} \\ N_{s\theta n} \\ N_{\theta n} \end{Bmatrix} = \overline{K} \begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} \tag{26}$$

$$\begin{Bmatrix} M_{sn} \\ M_{s\theta n} \\ M_{\theta n} \end{Bmatrix} = \widetilde{K} \begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} \tag{27}$$

where

$$\overline{K} = e^{x} \begin{bmatrix} \sin\alpha(-\mathcal{D}+\nu) & -\nu n & \nu\cos\alpha \\ \frac{1-\nu}{2}n & -\frac{1-\nu}{2}\sin\alpha(\mathcal{D}+1) & 0 \\ \sin\alpha(-\nu\mathcal{D}+1) & -n & \cos\alpha \end{bmatrix} \tag{28}$$

$$\widetilde{K} = e^{2x} \begin{bmatrix} 0 & 0 & \sin^2\alpha\left\{-\mathcal{D}^2-(1-\nu)\mathcal{D}\right\}+\nu n^2 \\ 0 & 0 & \frac{1-\nu}{2}n\sin\alpha(\mathcal{D}+1) \\ 0 & 0 & \sin^2\alpha\left\{-\nu\mathcal{D}^2+(1-\nu)\mathcal{D}\right\}+n^2 \end{bmatrix} \tag{29}$$

It might be noted from (29) that the couple resultants M's depend on transverse displacement w_n only, as a consequence of the simplification made in the stress-strain relations.

Substitution of (26) and (27) into (23) gives a matrix equation for displacements:

$$\overline{H}\begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} + \widetilde{H}\begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} = \Omega^2 \begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} \tag{30}$$

where $\overline{H} = \overline{F}\overline{K}$ and $\widetilde{H} = \widetilde{F}\widetilde{K}$ are (3×3) operator matrices governing the membrane and bending effects, respectively

$$\overline{H} = e^{2x}\begin{bmatrix} \sin^2 \alpha(-\mathcal{D}^2+1)+\frac{1-\nu}{2}n & -n\sin\alpha\left(\frac{1+\nu}{2}\mathcal{D}+\frac{3-\nu}{2}\right) & \sin\alpha\cos\alpha(\nu\mathcal{D}+1) \\ n\sin\alpha\left(\frac{1+\nu}{2}\mathcal{D}-\frac{3-\nu}{2}\right) & \frac{1-\nu}{2}\sin^2\alpha(-\mathcal{D}^2+1)+n^2 & -n\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha(\nu\mathcal{D}-1) & -n\cos\alpha & \cos^2\alpha \end{bmatrix} \tag{31}$$

$$\widetilde{H} = \epsilon e^{4x}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & n\cos\alpha(\sin^2\alpha\mathcal{D}^2-n^2) \\ 0 & 0 & [\sin^2\alpha(\mathcal{D}+2)^2-n^2](\sin^2\alpha\mathcal{D}^2-n^2) \end{bmatrix} \tag{32}$$

It is seen from (32) that the fourth derivative of w_n is involved in the bending effects. Therefore, the Fourier expansion of w_n must be termwise differentiable four times to ensure convergence of the Galerkin procedure. This requirement limits the direct application of Galerkin method to equation (30). For this reason, a calculation procedure based on equation (22) is recommended.

Membrane Solutions

It can be seen from (25) and (29) that the bending effects are proportional to (h^2/a^2). Therefore, for thin shells we can neglect the second term of (23) or (30) to find an approximate solution. Thus, from (23), we have

$$\overline{F}\begin{Bmatrix} N_{sn} \\ N_{s\theta n} \\ N_{\theta n} \end{Bmatrix} = \Omega^2 \begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} \tag{33}$$

which, combined with (26), gives

$$\overline{H}\begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} = \Omega^2 \begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} \tag{34}$$

and

$$\overline{J}\begin{Bmatrix} N_{sn} \\ N_{s\theta n} \\ N_{\theta n} \end{Bmatrix} = \Omega^2 \begin{Bmatrix} N_{sn} \\ N_{s\theta n} \\ N_{\theta n} \end{Bmatrix} \tag{35}$$

where $\overline{H}$ is given in (31), and $\overline{J} = \overline{K}\overline{F}$ is the (3×3) operator matrix:

$$\overline{J} = e^{2x}\sin^2 \alpha \begin{bmatrix} -\mathcal{D}^2 + \nu\mathcal{D} + (1-\nu) & -\dfrac{n}{\sin \alpha}\left[(1+\nu)\mathcal{D} + (1-3\nu)\right] & -(\mathcal{D}+1) + \dfrac{\nu(n^2+1)}{\sin^2 \alpha} \\ \dfrac{1-\nu}{2}\,\dfrac{n}{\sin \alpha}(\mathcal{D}-1) & -\dfrac{1-\nu}{2}\left(\mathcal{D}^2 - 4 - \dfrac{n^2}{\sin^2 \alpha}\right) & \dfrac{1-\nu}{2}\,\dfrac{n}{\sin \alpha}(\mathcal{D}+3) \\ -\nu\mathcal{D}^2 + \mathcal{D} - (1-\nu) & -\dfrac{n}{\sin \alpha}\left[(1+\nu)\mathcal{D} - (3-\nu)\right] & -\nu(\mathcal{D}+1) + \dfrac{n^2+1}{\sin^2 \alpha} \end{bmatrix} \tag{36}$$

Equation (34) can be used for clamped or simply supported edge conditions as discussed above, while (35) can be used to find solutions for extensional vibrations of free-free conical shells, which will be given here for illustration.

The boundary conditions for free edges are $N_{sn} = N_{s\theta n} = 0$, at $x = 0, L$, which are satisfied by taking

$$N_{sn} = \sum_{m=1}^{M_1} a_m \sin \frac{m\pi x}{L} \tag{37a}$$

$$N_{s\theta n} = \sum_{m=1}^{M_2} b_m \sin \frac{m\pi x}{L} \tag{37b}$$

$$N_{\theta n} = \frac{c_0}{2} + \sum_{m=1}^{M_3} c_m \cos \frac{m\pi x}{L} \tag{37c}$$

It might be remarked here that the constant term in the cosine series must be included to make the set of coordinate functions complete. In fact, the constant c_0 will be the dominating term in a specific mode to be designated as ring mode. Substitution of (37) into (36) and application of the Galerkin procedure give $(M_1 + M_2 + M_3 + 1)$ algebraic equations for the coefficients, a's, b's and c's. Thus,

$$\begin{bmatrix} [\bar{J}^{11}_{km}] & [\bar{J}^{12}_{km}] & [\bar{J}^{13}_{km}] \\ [\bar{J}^{21}_{km}] & [\bar{J}^{22}_{km}] & [\bar{J}^{23}_{km}] \\ [\bar{J}^{31}_{km}] & [\bar{J}^{32}_{km}] & [\bar{J}^{33}_{km}] \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ b_1 \\ \vdots \\ c_0 \\ \vdots \end{Bmatrix} = \Omega^2 \begin{Bmatrix} a_1 \\ \vdots \\ b_1 \\ \vdots \\ c_0 \\ \vdots \end{Bmatrix} \tag{38}$$

where $[\bar{J}^{ij}_{km}]$, $i, j = 1, 2, 3$, are partition matrices whose elements are given by the following formulas

$$\left.\begin{aligned} \bar{J}^{11}_{km} &= \sin^2 \alpha \left[\left(\frac{m\pi}{L}\right)^2 P_{km} + \nu \left(\frac{m\pi}{L}\right) Q_{km} + (1 - \nu) P_{km}\right] && m = 1 \sim M_1 \\ \bar{J}^{12}_{km} &= - n \sin \alpha \left[(1 - \nu) \left(\frac{m\pi}{L}\right) Q_{km} + (1 - 3\nu) P_{km}\right] && m = 1 \sim M_2 \\ \bar{J}^{13}_{km} &= \left(\frac{m\pi}{L}\right) \sin^2 \alpha\, P_{km} + (\nu n^2 + \nu - \sin^2 \alpha) Q_{km} && m = 0 \sim M_3 \end{aligned}\right\} k = 1 \sim M_1 \tag{39a}$$

$$\left.\begin{aligned} \bar{J}^{21}_{km} &= \frac{1 - \nu}{2} n \sin \alpha \left[\left(\frac{m\pi}{L}\right) Q_{km} - P_{km}\right] && m = 1 \sim M_1 \\ \bar{J}^{22}_{km} &= \frac{1 - \nu}{2} \left[\left(\frac{m\pi}{L}\right)^2 + 4 \sin^2 \alpha + n^2\right] P_{km} && m = 1 \sim M_2 \\ \bar{J}^{23}_{km} &= - \frac{1 - \nu}{2} n \sin \alpha \left[\left(\frac{m\pi}{L}\right) P_{km} - 3Q_{km}\right] && m = 0 \sim M_3 \end{aligned}\right\} k = 1 \sim M_2 \tag{39b}$$

$$\bar{J}^{31}_{km} = \sin^2 a \left[\nu \left(\frac{m\pi}{L}\right)^2 Q_{mk} + \left(\frac{m\pi}{L}\right) R_{km} - (1 - \nu) Q_{mk}\right] \qquad m = 1 \sim M_1$$

$$\bar{J}^{32}_{km} = - n \sin a \left[(1 + \nu)\left(\frac{m\pi}{L}\right) R_{km} - (3 - \nu) Q_{mk}\right] \qquad m = 1 \sim M_2 \qquad k = 0 \sim M_3$$

$$\bar{J}^{33}_{km} = \nu \sin^2 a \left(\frac{m\pi}{L}\right) Q_{mk} + (n^2 + 1 - \nu \sin^2 a) R_{km} \qquad m = 0 \sim M_3 \tag{39c}$$

$$P_{km} = \frac{2}{L} \int_0^L e^{2x} \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx$$

$$= \left[\frac{2L}{4L^2 + (k - m)^2 \pi^2} - \frac{2L}{4L^2 + (k + m)^2 \pi^2}\right] [e^{2L} \cos (k + m) \pi - 1]$$

$$Q_{km} = \frac{2}{L} \int_0^L e^{2x} \sin \frac{k\pi x}{L} \cos \frac{m\pi x}{L} \, dx$$

$$= - \left[\frac{(k - m) \pi}{4L^2 + (k - m)^2 \pi^2} + \frac{(k + m) \pi}{4L^2 + (k + m)^2 \pi^2}\right] [e^{2L} \cos (k + m) \pi - 1]$$

$$m \geqslant 1$$

$$R_{km} = \frac{2}{L} \int_0^L e^{2x} \cos \frac{k\pi x}{L} \cos \frac{m\pi x}{L} \, dx$$

$$= \left[\frac{2L}{4L^2 + (k - m)^2 \pi^2} + \frac{2L}{4L^2 + (k + m)^2 \pi^2}\right] [e^{2L} \cos (k + m) \pi - 1]$$

$$m \geqslant 1$$

$$Q_{k0} = \frac{1}{L} \int_0^L e^{2x} \sin \frac{k\pi x}{L} \, dx = - \frac{k\pi}{4L^2 + k^2\pi^2} [e^{2L} \cos (k\pi) - 1]$$

$$R_{k0} = \frac{1}{L}\int_0^L e^{2x} \cos \frac{k\pi x}{L}\, dx = \frac{2L}{4L^2 + k^2\pi^2}\left[e^{2L} \cos k\pi - 1\right]$$

NUMERICAL RESULTS AND DISCUSSION

It is seen that the coefficients $\bar{J}^{ij}_{km}$ depend on four parameters, namely, the Poisson's ratio ν, the semivertex angle α, the completeness parameter s_2/s_1, and the circumferential wave number n. Poisson's ratio is taken as 0.3. For a set of assigned values of the other three parameters and properly chosen integers M_1, M_2, M_3, the formulas (39) give numerical values for the coefficient matrix of (38). As an example, we set $\alpha = 30°$, $s_2/s_1 = 2.0$, $n = 1$, $M_1 = M_2 = 3$, and $M_3 = 5$, then the formulas (39) generate a (12×12) matrix. The eigenvalues and associated eigenvectors were calculated by a Jacobi-like method* of diagonalizing matrices developed by Eberlein (Ref. 22). The resulting frequencies and eigenvectors are given in Table 1, listed in the same order as they appear in the output diagonal matrix. It is seen that the eigenvalues appear in an interesting arrangement and may be classified into three groups. The first three modes may be termed as "longitudinal modes" in which a_1, a_2, and a_3 predominate in turn, thus the vibrations are mainly associated with the meridional stress N_s. The fourth, fifth and sixth modes may be called "shear modes" (or rather "torsional modes" for $n = 0$, the axisymmetric case). The remaining modes in which N_θ predominates are mainly associated with the transverse motion and may be termed as "transverse modes." From Table 1, it is as expected that the cosine series (37c) converges slower than the two sine series (37a, b) and requires six terms to make the first two transverse modes (Nos. 7 and 8) have satisfactory accuracy. It should be remarked here that the special arrangement and automatic classification of the three types of modes appear only for a certain range of the parameters α, s_2/s_1, and n, in which the coupling effects are weak. It should also be mentioned that, although twelve modes were obtained from the solution, only the first longitudinal mode, No. 1, the first shear mode, No. 4, and the first two transverse modes, Nos. 7, 8, have good accuracy. The last two eigenvalues were not shown in Table 1 because they appear as (2×2) diagonal block in the output diagonal matrix, which means (Ref. 22) the eleventh and twelfth eigenvalues are conjugate

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*The subrouting program used in the calculations is based on one prepared by Eberlein, Computing Center, University of Rochester, N. Y. For details of the method, see Ref. 22.

TABLE 1. NATURAL FREQUENCIES AND FOURIER COEFFICIENTS OF MODE FUNCTIONS, N_{sn}, $N_{s\theta n}$, $N_{\theta n}$, OF FREE-FREE CONICAL SHELLS WITH $\alpha = 30°$, $s_2/s_1 = 2.0$, $n = 1$

No.	Ω	a_1	a_2	a_3	b_1	b_2	b_3	c_0	c_1	c_2	c_3	c_4	c_5
1	3.084	.904	.050	-.015	.023	-.195	-.029	.370	.116	.078	-.016	-.003	-.017
2	6.903	-.267	.721	.146	.021	-.223	.464	-.017	.063	.281	-.337	.040	-.032
3	10.279	.077	-.396	.903	.013	-.024	-.027	.147	-.122	.172	.075	-.162	.056
4	2.774	.175	.080	.015	.371	.117	-.012	.898	-.547	-.033	-.050	.002	-.044
5	4.308	.499	.046	.091	-.290	.663	.217	-.096	.458	-.478	-.097	-.071	.054
6	5.937	.289	-.923	-.177	.101	-.215	.507	.185	-.318	-.007	-.136	.111	.009
7	1.585	-.137	-.026	-.013	-.228	-.032	.013	1.271	.278	-.080	.130	.020	.025
8	0.850	-.141	-.054	-.035	.244	.129	.078	-.238	.445	.591	.597	.102	.068
9	1.221	.036	-.127	.045	-.254	.258	-.061	.283	-.684	1.048	-.294	.000	.036
10	1.111	.079	-.064	-.042	-.254	.085	.098	.195	-.553	.207	.694	.118	.070

complex numbers as a result of the series truncation. It is found from computation experience that, if unequal integers are assigned to M_1, M_2, and M_3, there will be a danger of obtaining complex eigenvalues.

Since the bending rigidity has negligibly small effects on lower modes of axisymmetric vibrations of free-free conical shells, we will investigate this case in detail by applying equation (38). From (36) it is seen that, when $n = 0$, we have

$$\bar{J}^{12} = \bar{J}^{21} = \bar{J}^{23} = \bar{J}^{32} = 0$$

Therefore, the second equation of (35) is uncoupled from the other two; that is, the torsional modes are completely independent from the coupled longitudinal and transverse modes. From (39) we can see that the torsional modes are given by the submatrix $[\bar{J}^{22}_{km}]$. Since we are now interested in the nontorsional axisymmetric vibrations of free-free conical shells, we set $\nu = 0.3$, $n = 0$, $M_1 = M_3 = 5$, and $M_2 = 0$ in formulas (39). The frequency now depends on α and s_2/s_1 only.

To investigate the dependence of frequency on α, we fix $s_2/s_1 = 2.0$, and take a series of values for α, ($\alpha = 3°$, $5°$, $10°$, $15°$, $20°$, $30°$, $45°$, $60°$, $75°$, and $85°$), then calculate the frequencies from the (11×11) matrices. The results are plotted in Figure 3. It was found that, for $\alpha > 15°$, the frequencies appear as two groups in the output diagonal matrices, and the frequencies of longitudinal modes are always higher than those of transverse modes. But for $\alpha < 15°$, the frequencies of the two groups become interspersed, and no pronounced transverse modes exist. This phenomenon will be referred to as "strong coupling." For different values of the completeness parameter s_2/s_1, different curves similar to Figure 3 will be obtained. Figure 4 represents the case of $s_2/s_1 = 4.0$, in which strong coupling occurs in the region $0 < \alpha < 45°$.

It might be noted from Figures 3 and 4 that, while the frequency spectra of longitudinal modes extend to infinity, the frequencies of higher transverse modes are spaced in a finite interval, the shaded region. This result is, as expected, the limiting case when the shell thickness tends to zero. For real shells with even small thickness, the frequencies of higher transverse modes in the shaded area and of all transverse modes in some neighborhood of $\alpha = 90°$ are expected to be increased considerably by the bending rigidity.

Another interesting result from the above calculations is that the frequency associated with the constant term c_0 of the cosine series (37c) tends to a finite value as $\alpha \rightarrow 90°$. This particular type of vibration may be called a "ring mode" since the entire shell vibrates without a nodal circle. The

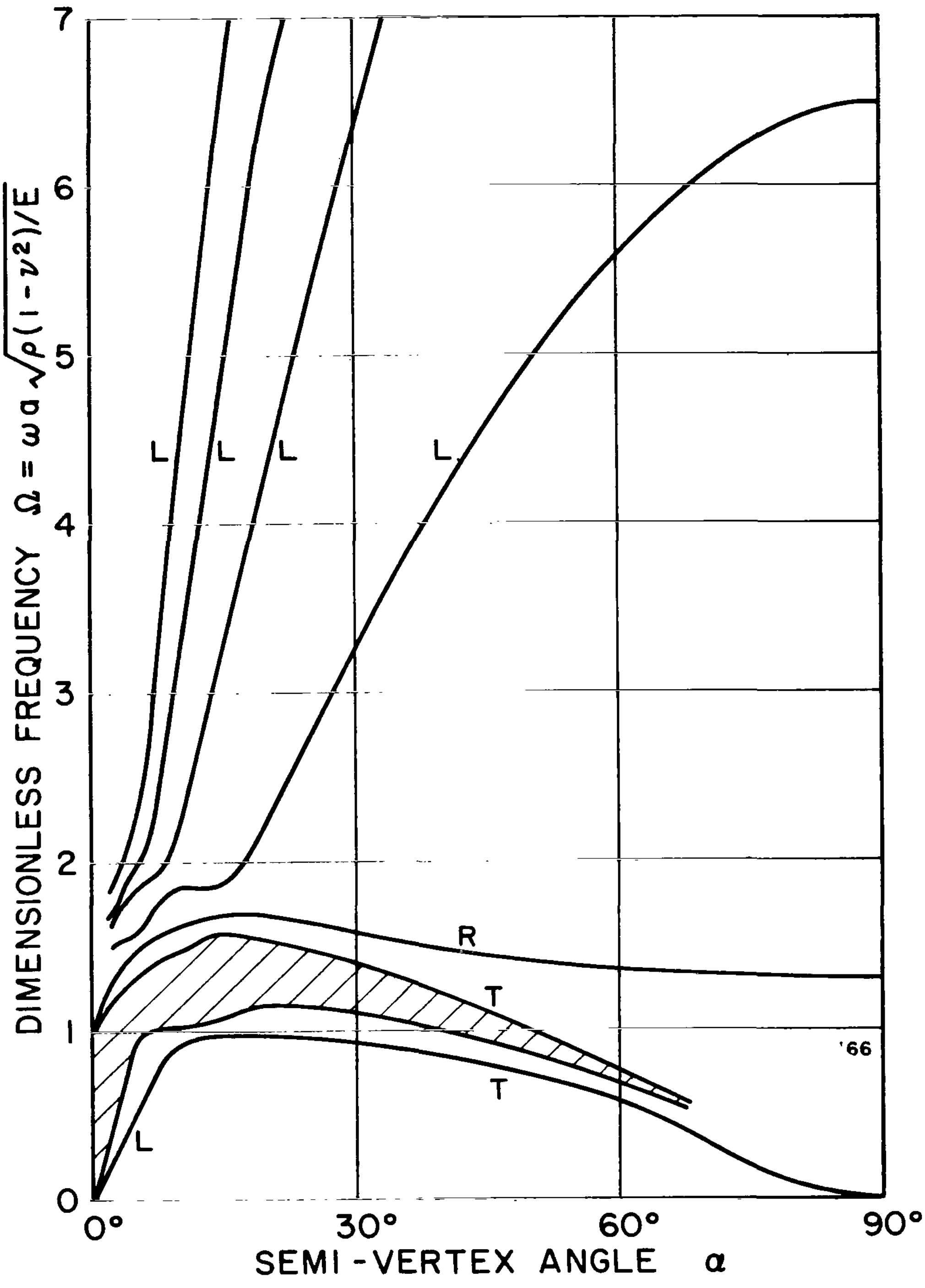

FIGURE 3. VARIATION OF FREQUENCY WITH RESPECT TO SEMIVERTEX ANGLE α FOR FREE-FREE CONICAL SHELL WITH $s_2/s_1 = 2.0$, $n = 0$

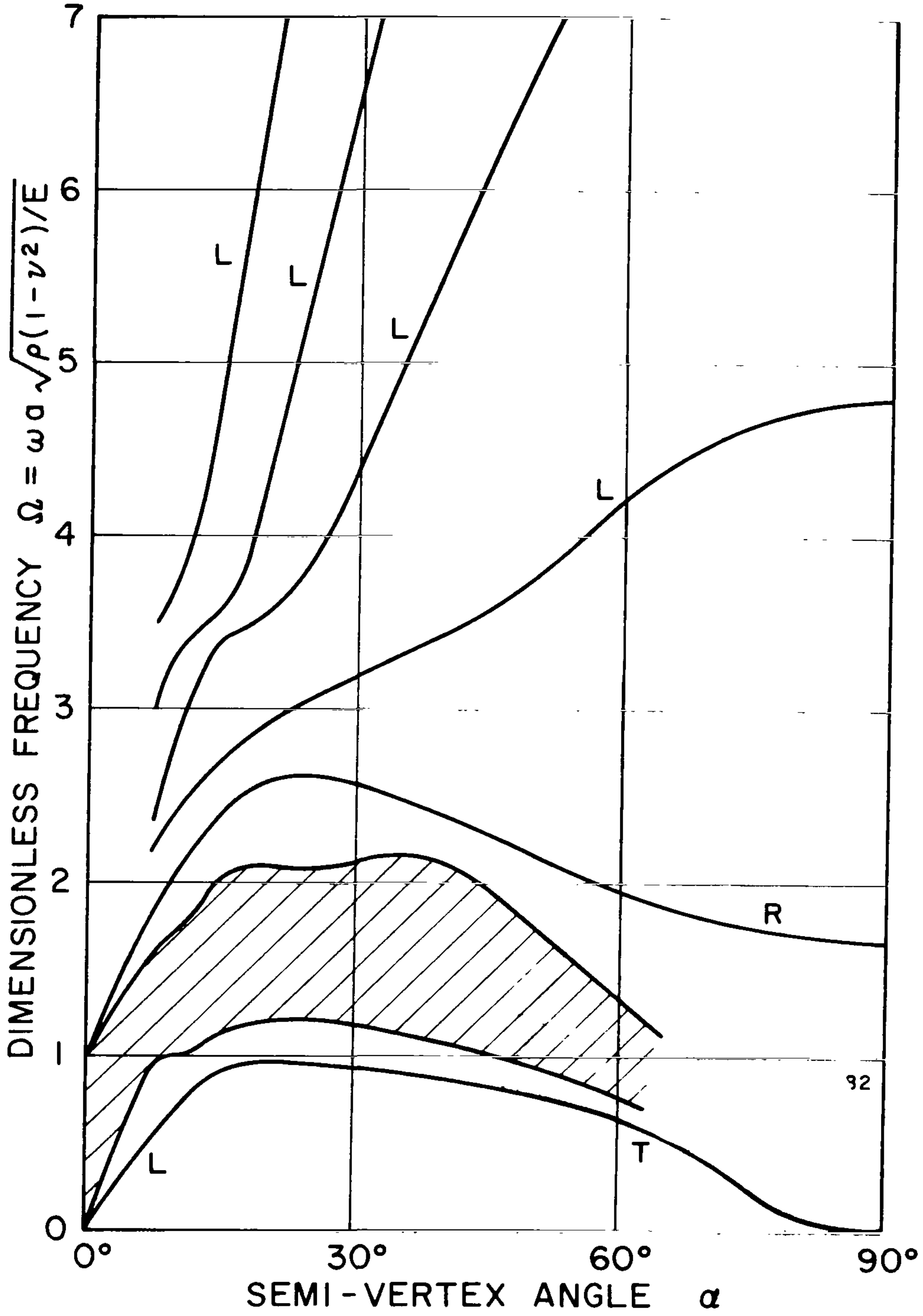

FIGURE 4. VARIATION OF FREQUENCY WITH RESPECT TO SEMIVERTEX ANGLE α FOR FREE-FREE CONICAL SHELL WITH $s_2/s_1 = 4.0$, $n = 0$

different types of modes are labeled in Figures 3 and 4 by L for longitudinal, T for transverse, and R for ring modes.

The dependence of the frequency on the completeness parameter s_2/s_1 was investigated in a similar manner. Setting $\alpha = 15°$ and taken successively $s_2/s_1 = 1.1$, 1.25, 1.75, 2.0, 2.25, 2.5, 3.0, 4.0, and 5.0, we calculate frequencies from the (11×11) matrices. The results are plotted in Figure 5. It is found that the two types of modes have weak coupling for $s_2/s_1 < 2$ and strong coupling for $s_2/s_1 > 2$. It is seen that the frequencies have a very interesting and pronounced steplike variation with s_2/s_1, typical of this type of problem. Another interesting conclusion which may be drawn from Figure 5 is that, for $\alpha = 15°$, the lowest frequency is nearly independent of the completeness of the conical shell.

From the above numerical analysis, Figure 6 was contrived and summarizes graphically the influence of the meridional stress resultant N_s on the transverse modes of axisymmetric vibrations of free-free conical shells. While strong coupling will always exist for conical shells in the upper-left region in Figure 6, the lower-right region includes shells with weak coupling effects which ensure the existence of pronounced transverse modes. The coupling effects can be more easily seen from the mode functions. Some typical truncated series representations of the mode functions are shown in Figures 7 through 10. Figures 7 and 8 show the mode functions of shells in the weak coupling region. It is seen that, in transverse modes, N_{sn} is small compared to $N_{\theta n}$. Figures 9 and 10 show the mode functions of two shells belonging to the strong coupling region. The modes can no longer be classified, and N_{sn} and $N_{\theta n}$ are of the same order of magnitude.

CONCLUSIONS

The matrix method presented in this paper provides a systematic way of calculating natural frequencies and mode functions of truncated conical shells with the same type of boundary condition at both edges. It is not applicable to mixed boundary value problems, such as fixed-free conical shells, because of the same difficulty as will be encountered in applying the Rayleigh-Ritz method, namely, the difficulty of choosing coordinate functions to satisfy all boundary conditions. However, whenever applicable, it provides an efficient calculation procedure which preserves the mathematical characteristics of the vibration problem as an eigenvalue problem.

Finally, it may be remarked that the numerical results obtained in the present work confirm the conclusion observed previously by many others, that a one- or two-term Rayleigh-Ritz approximation cannot yield satisfactory results except for slightly tapered, short conical shells. The calculations also indicate that, in most cases, more than five terms are needed to give a satisfactory series representation of the mode functions.

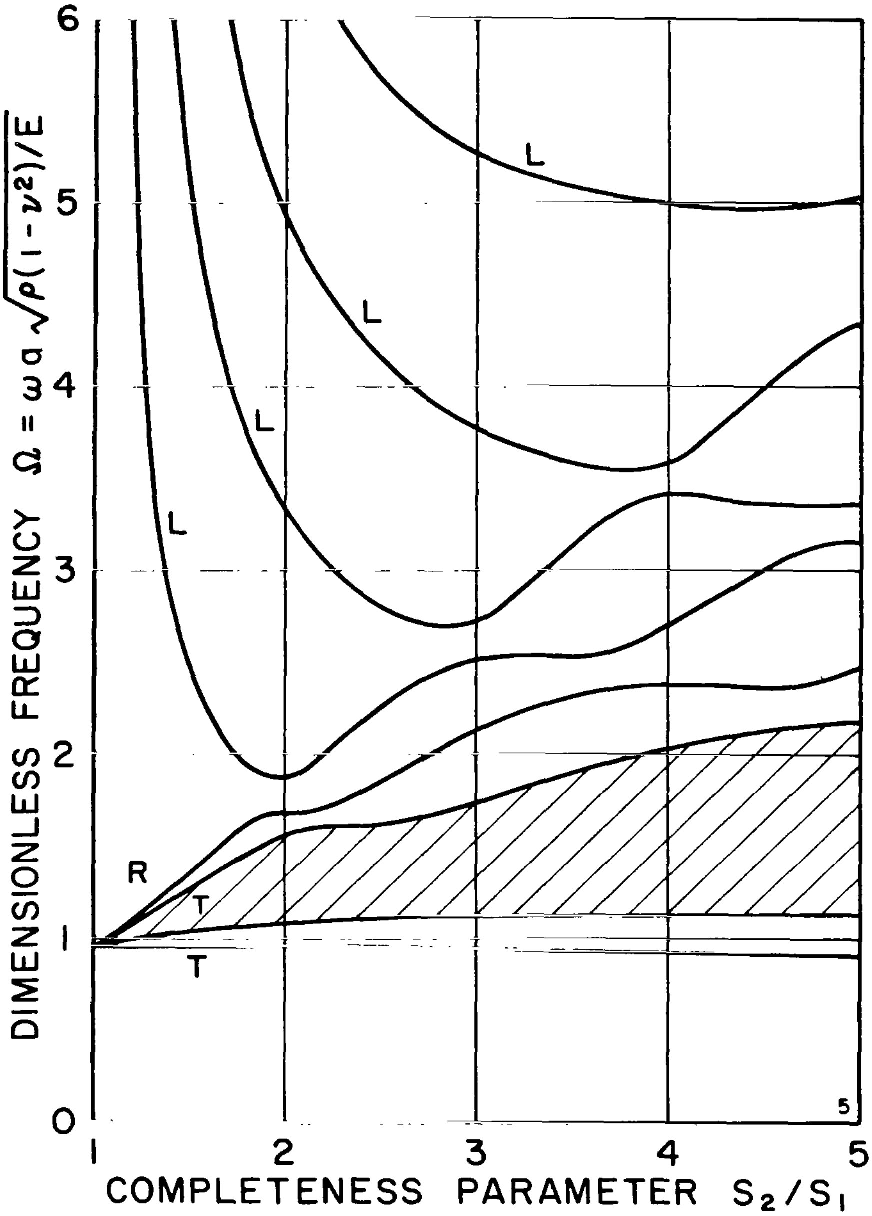

FIGURE 5. VARIATION OF FREQUENCY WITH RESPECT TO COMPLETENESS PARAMETER s_2/s_1 FOR FREE-FREE CONICAL SHELL WITH α = 15°, n = 0

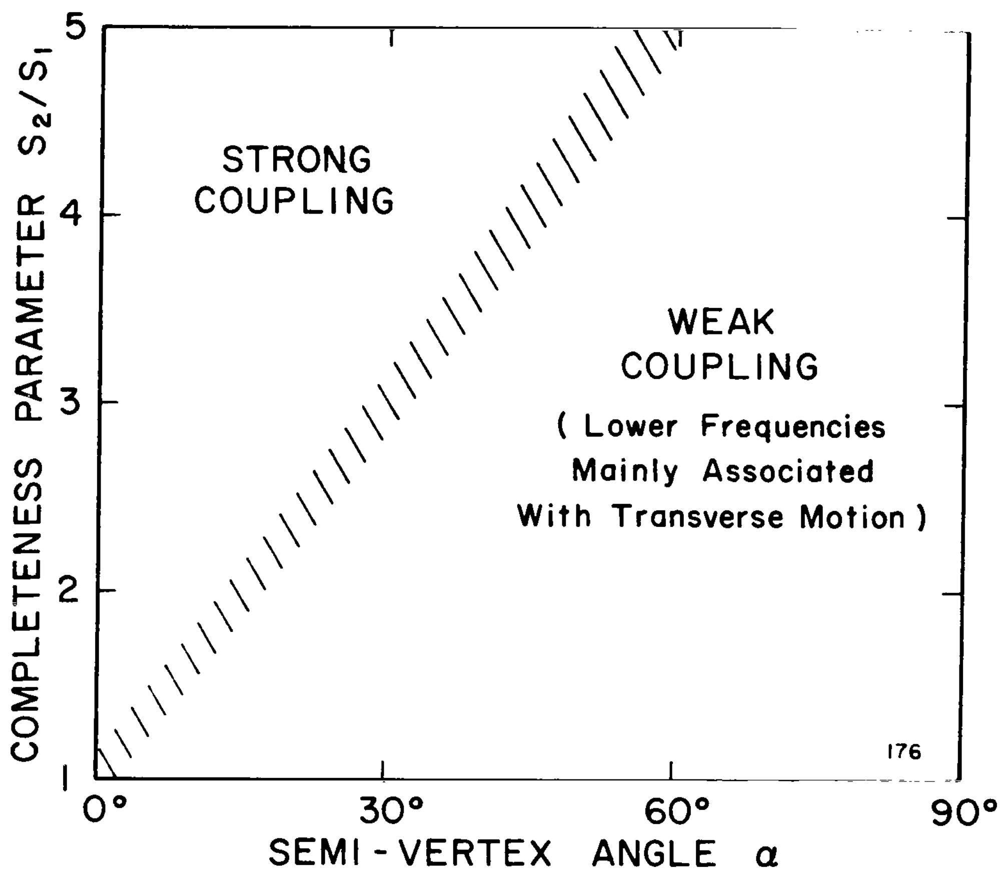

FIGURE 6. INFLUENCE OF MERIDIONAL STRESS RESULTANT N_s ON TRANSVERSE MODES OF AXISYMMETRIC VIBRATIONS OF FREE-FREE CONICAL SHELLS

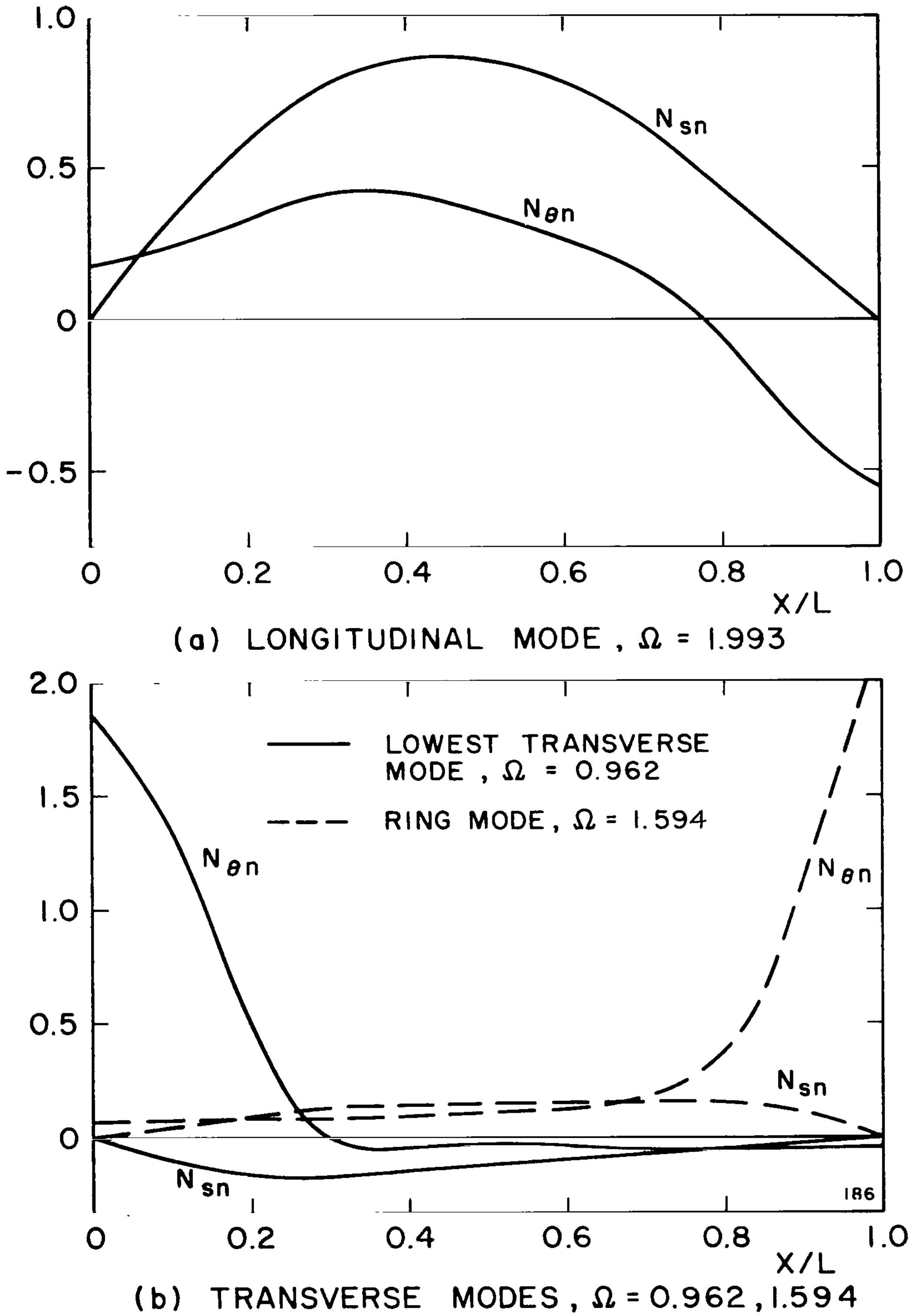

FIGURE 7. MODE FUNCTIONS OF FREE-FREE CONICAL SHELL WITH $\alpha = 15°$, $s_2/s_1 = 1.75$, $n = 0$

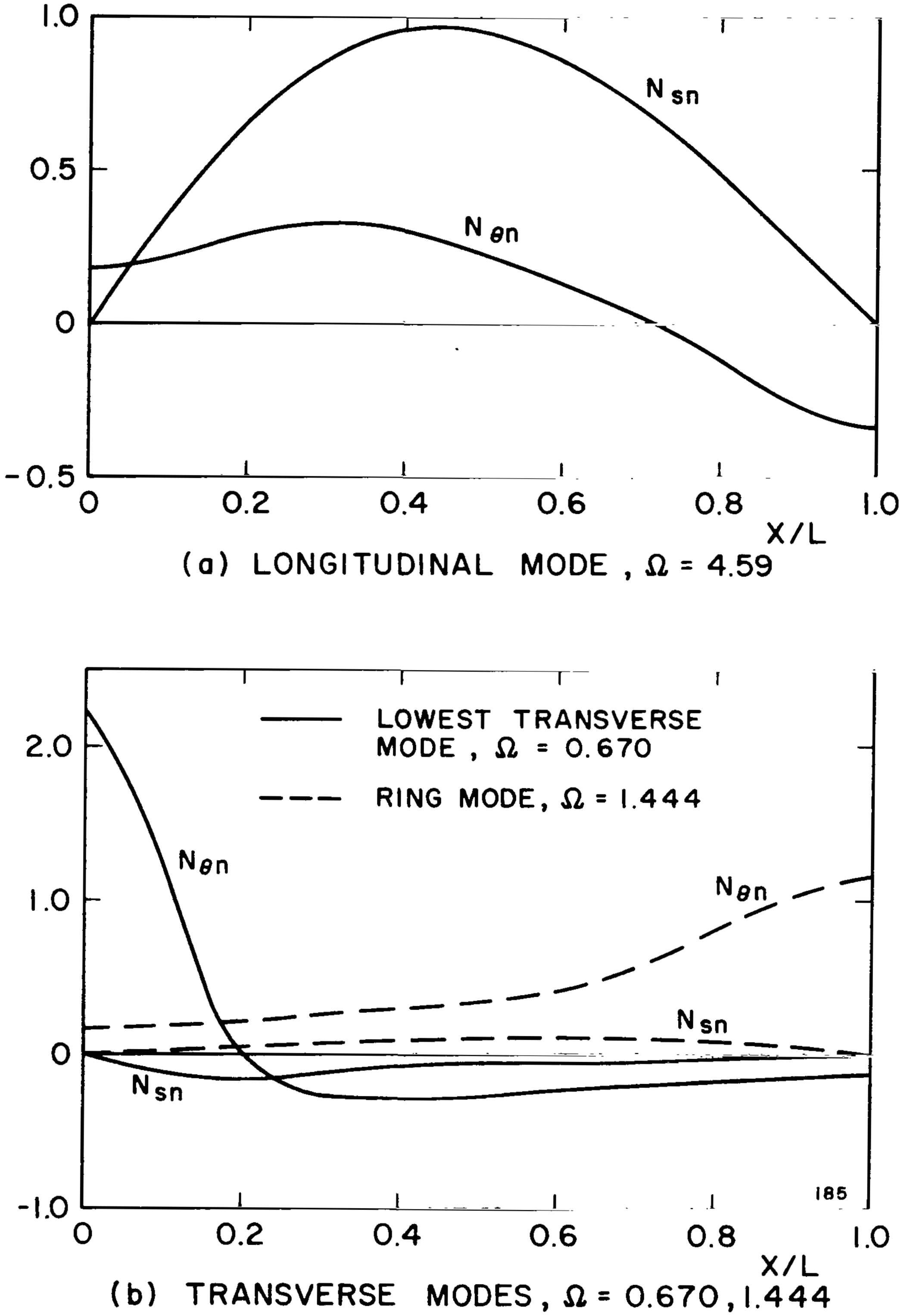

FIGURE 8. MODE FUNCTIONS OF FREE-FREE CONICAL SHELL WITH $\alpha = 45°$, $s_2/s_1 = 2.0$, $n = 0$

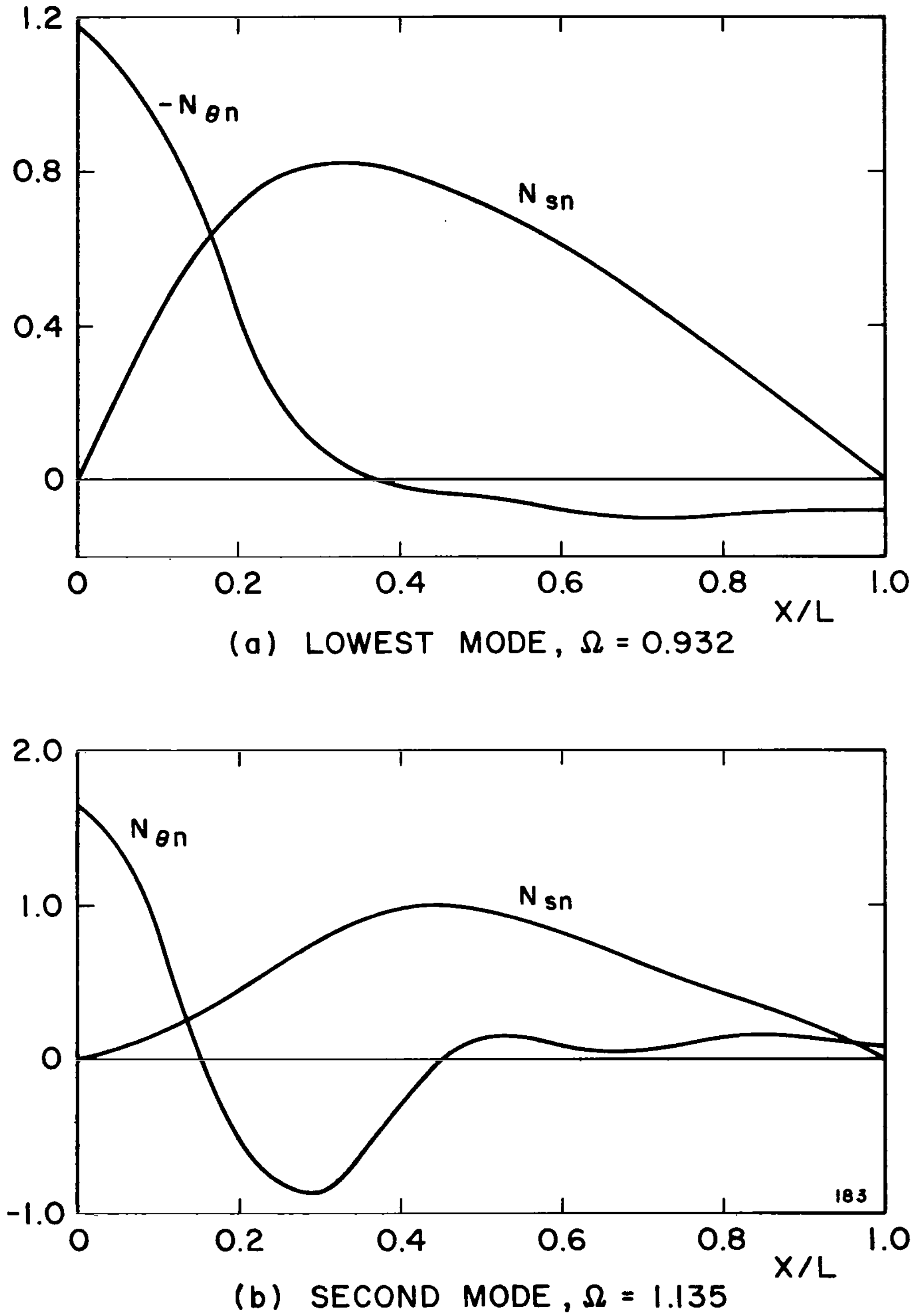

FIGURE 9. MODE FUNCTIONS OF FREE-FREE CONICAL SHELL WITH $\alpha = 15°$, $s_2/s_1 = 3.0$, $n = 0$

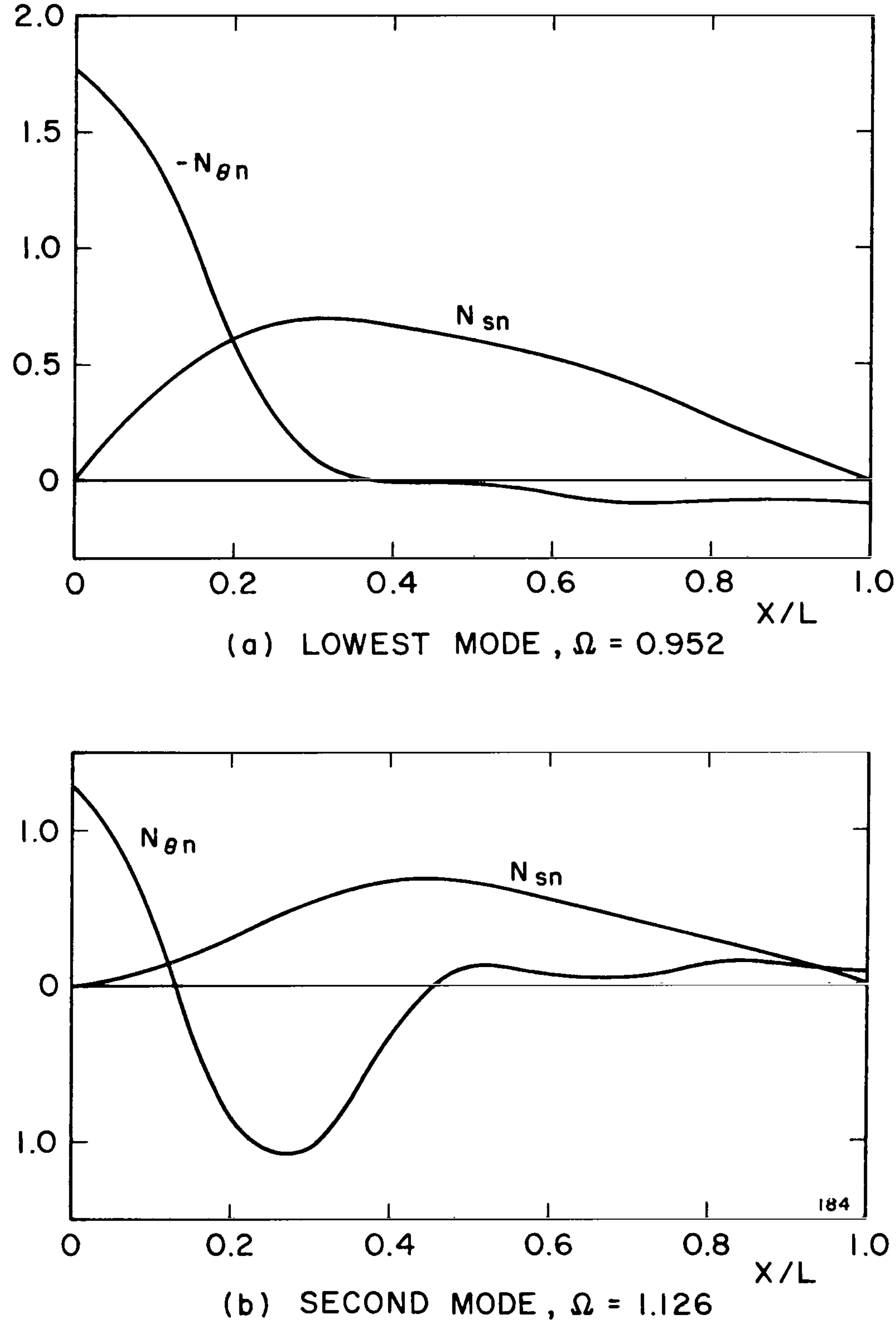

FIGURE 10. MODE FUNCTIONS OF FREE-FREE CONICAL SHELL WITH $\alpha = 15°$, $s_2/s_1 = 4.0$, $n = 0$

APPENDIX I

The elements of the operator matrix H

$$H^{11} = e^{2x}\left[-\sin^2\alpha\,(\mathcal{D}^2 - 1) + \frac{1-\nu}{2}\,n^2\right]$$

$$H^{12} = -\,n\sin\alpha\,e^{2x}\left[\frac{1+\nu}{2}\mathcal{D} + \frac{3-\nu}{2}\right]$$

$$H^{13} = \sin\alpha\cos\alpha\,e^{2x}\,(\nu\mathcal{D} + 1)$$

$$H^{14} = H^{15} = 0$$

$$H^{21} = n\sin\alpha\,e^{2x}\left[\frac{1+\nu}{2}\mathcal{D} - \frac{3-\nu}{2}\right]$$

$$H^{22} = e^{2x}\left[-\frac{1-\nu}{2}\sin^2\alpha\,(\mathcal{D}^2 - 1) + n^2\right]$$

$$H^{23} = -(1 + k_s)\,n\cos\alpha\,e^{2x}$$

$$H^{24} = 0$$

$$H^{25} = -\,k_s\cos\alpha\,e^{x}$$

$$H^{31} = -\,\sin\alpha\cos\alpha\,e^{2x}\,(\nu\mathcal{D} - 1)$$

$$H^{32} = -\,n\cos\alpha\,e^{2x}$$

$$H^{33} = e^{2x}\,[\,-k_s\sin^2\alpha\,\mathcal{D}^2 + k_s n^2 + \cos^2\alpha]$$

$$H^{34} = k_s\sin\alpha\,e^{x}\,(\mathcal{D} - 1)$$

$$H^{35} = k_s n e^x$$

$$H^{41} = H^{42} = 0$$

$$H^{43} = -(k_s/\epsilon) \sin\alpha \, e^x \mathcal{D}$$

$$H^{44} = e^{2x}\left[-\sin^2\alpha \,(\mathcal{D}^2 - 1) + \frac{1-\nu}{2} n \sin\alpha\right] + (k_s/\epsilon)$$

$$H^{45} = -n \sin\alpha \, e^{2x}\left[\frac{1+\nu}{2}\mathcal{D} + \frac{3-\nu}{2}\right]$$

$$H^{51} = H^{52} = 0$$

$$H^{53} = (k_s/\epsilon)\, n\, e^x$$

$$H^{54} = n \sin\alpha \, e^{2x}\left[\frac{1+\nu}{2}\mathcal{D} - \frac{3-\nu}{2}\right]$$

$$H^{55} = -\frac{1-\nu}{2}\sin^2\alpha \, e^{2x}(\mathcal{D}^2 - 1) + n^2 e^{2x} + (k_s/\epsilon)$$

The elements of the operator matrix J

$$J^{11} = J^{44} = -e^{2x}\sin^2\alpha\,[\mathcal{D}^2 - \nu\mathcal{D} - (1-\nu)]$$

$$J^{12} = J^{45} = -n \sin\alpha \, e^{2x}[(1+\nu)\mathcal{D} + (1-3\nu)]$$

$$J^{13} = e^{2x}[-\sin^2\alpha\,(\mathcal{D}+1) + \nu(1+n^2)]$$

$$J^{14} = J^{15} = J^{16} = 0$$

$$J^{17} = \nu \epsilon \sin \alpha \cos \alpha \, e^{2x} (\mathcal{D} - 1)$$

$$J^{18} = 2\nu \epsilon n \cos \alpha \, e^{2x}$$

$$J^{21} = J^{54} = \frac{1 - \nu}{2} n \sin \alpha \, e^{2x} (\mathcal{D} - 1)$$

$$J^{22} = J^{55} = - \frac{1 - \nu}{2} e^{2x} [\sin^2 \alpha (\mathcal{D}^2 - 4) - n^2]$$

$$J^{23} = J^{56} = \frac{1 - \nu}{2} n \sin \alpha \, e^{2x} (\mathcal{D} + 3)$$

$$J^{24} = J^{25} = J^{26} = J^{27} = 0$$

$$J^{28} = \frac{1 - \nu}{2} \epsilon \sin \alpha \cos \alpha \, e^{2x} (\mathcal{D} + 2)$$

$$J^{31} = J^{64} = - \sin^2 \alpha \, e^{2x} [\nu \mathcal{D}^2 - \mathcal{D} + (1 - \nu)]$$

$$J^{32} = J^{65} = - n \sin \alpha \, e^{2x} [(1 + \nu) \mathcal{D} - (3 - \nu)]$$

$$J^{33} = e^{2x} [- \nu \sin^2 \alpha (\mathcal{D} + 1) + 1 + n^2]$$

$$J^{34} = J^{35} = J^{36} = 0$$

$$J^{37} = \epsilon \sin \alpha \cos \alpha \, e^{2x} (\mathcal{D} - 1)$$

$$J^{38} = 2\epsilon n \cos \alpha \, e^{2x}$$

$$J^{41} = J^{42} = J^{43} = 0$$

$$J^{46} = e^{2x} [-\sin^2 a (\mathcal{D} + 1) + \nu (\sin^2 a + n^2)]$$

$$J^{47} = - e^x \sin a (\mathcal{D} - \nu)$$

$$J^{48} = - \nu n e^x$$

$$J^{51} = J^{52} = J^{53} = 0$$

$$J^{57} = \frac{1 - \nu}{2} n e^x$$

$$J^{58} = - \frac{1 - \nu}{2} \sin a \, e^x (\mathcal{D} + 1)$$

$$J^{61} = J^{62} = J^{63} = 0$$

$$J^{66} = - e^{2x} [\nu \sin^2 a (\mathcal{D} + 1) - \sin^2 a - n^2]$$

$$J^{67} = - \sin a \, e^x (\nu \mathcal{D} - 1)$$

$$J^{68} = - n e^x$$

$$J^{71} = J^{72} = 0$$

$$J^{73} = - (k_s / \epsilon) \sin a \cos a \, e^{2x} (\mathcal{D} + 1)$$

$$J^{74} = (k_s / \epsilon) \sin a \, e^x (\mathcal{D} - 1)$$

$$J^{75} = (k_s / \epsilon) n e^x$$

$$J^{76} = (k_s / \epsilon) \sin a \, e^x$$

$$J^{77} = - k_s \sin^2 \alpha \, e^{2x} (\mathscr{D}^2 - 1) + (k_s/\epsilon)$$

$$J^{78} = - k_s n \sin \alpha \, e^{2x} (\mathscr{D} + 1)$$

$$J^{81} = J^{82} = 0$$

$$J^{83} = (k_s/\epsilon) \, n \cos \alpha$$

$$J^{84} = 0$$

$$J^{85} = (k_s/\epsilon) \sin \alpha \, e^{x} (\mathscr{D} - 2)$$

$$J^{86} = - (k_s/\epsilon) \, n e^{x}$$

$$J^{87} = k_s n \sin \alpha \, e^{2x} (\mathscr{D} - 1)$$

$$J^{88} = k_s n^2 e^{2x} + (k_s/\epsilon)$$

APPENDIX II

The elements of the operator matrix H^*: H^{*11}, H^{*12}, H^{*13}, H^{*14}, H^{*21}, H^{*22}, H^{*31}, H^{*32}, H^{*41}, H^{*42}, and H^{*44}, are the same as corresponding elements H^{ij} in Appendix I.

$$H^{*23} = -n \cos \alpha e^{2x} + \epsilon n \cos \alpha\, e^{4x} \left[\frac{1-\nu}{2} \sin^2 \alpha\, (\mathcal{D}^2 + 2\mathcal{D}) - n^2\right]$$

$$H^{*24} = \epsilon n \sin \alpha \cos \alpha\, e^{3x} \left[\frac{1+\nu}{2} \mathcal{D} - \frac{3-\nu}{2}\right]$$

$$H^{*33} = e^{2x} [-k_s \sin^2 \alpha \mathcal{D}^2 + \cos^2 \alpha]$$

$$- \epsilon n^2 e^{4x} \left[\frac{1-\nu}{2} \sin^2 \alpha\, (\mathcal{D}^2 + 2\mathcal{D}) - n^2\right]$$

$$H^{*34} = k_s \sin \alpha\, e^{x} (\mathcal{D} - 1) - \epsilon n^2 \sin \alpha\, e^{3x} \left[\frac{1+\nu}{2} \mathcal{D} - \frac{3-\nu}{2}\right]$$

$$H^{*43} = -(k_s/\epsilon) \sin \alpha\, e^{x} \mathcal{D} + n^2 \sin \alpha\, e^{3x} \left[\frac{1+\nu}{2} \mathcal{D} + 2\right]$$

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